Introduction to the Theory of *Analytic Functions* of *Several Complex Variables*

by

B. A. Fuks

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Theory of

ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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ВВЕДЕНИЕ В ТЕОРИЮ
АНАЛИТИЧЕСКИХ ФУНКЦИЙ
МНОГИХ КОМПЛЕКСНЫХ
ПЕРЕМЕННЫХ

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PREFACE

The present volume has five chapters. The first chapter deals with the fundamental properties of analytic functions in the space of several complex variables, and the second chapter with the properties of analytic functions in covering regions over a suitable space. These two chapters may be considered as a textbook for readers who are looking for basic information, in as elementary a form as possible, about the theory of functions of several complex variables.

The next three chapters deal respectively with complex spaces, integral representations of functions of several complex variables, and functions meromorphic in the whole space. They are independent[1)] of one another in content but each of them makes a great deal of use of the material of the first two chapters. In contradistinction to the first two chapters, the last three are to a great extent in the nature of a survey. These chapters may serve as an introduction to the current technical literature on the various branches of the theory of functions.

The actual exposition itself is preceded by an introductory essay giving the most frequently used information from closely related mathematical disciplines. It is recommended that the reader refer to this essay whenever he finds it necessary.

The present book constitutes the first part of a second edition, considerably revised and enlarged, of the author's book *Theory of analytic functions of several complex variables* published in 1948. The second part, which is to appear soon after the first, will discuss a number of special chapters in the theory of functions.

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1) One exception is § 20 of Chapter IV, in which essential use is made of subsection 1, § 14 of Chapter III.

At the request of the author the first draft of the text of subsections 1–3, §23, dealing with integral representations in n-circular regions, was written by L. A. Aĭzenberg, subsections 4–6, §23, dealing with integral representations in tubular regions, by S. G. Gindikin, and section 26, dealing with methods of characterizing the growth of entire functions, by L. I. Ronkin. These sections contain a number of new results, which are due to the above mentioned persons and are introduced here, as a rule, without reference to the original articles.

An exposition of a number of original results referring to integral representations was kindly placed at my disposal by A. A. Temljakov.

I am also indebted to L. A. Aĭzenberg and D. B. Fuks, who looked over the entire text while it was being prepared for the press and gave me valuable advice.

To all the above persons I wish to express my profound gratitude.

Many sections of this book were first presented to the seminar on the theory of analytic functions at the University of Moscow. I wish to take this opportunity of thanking the members of the seminar, and several other mathematicians, who looked over various parts of the book and sent me their suggestions.

B. Fuks

March 1961

Translators' note: The present translation was prepared with the active assistance of the original author, Professor B. A. Fuks, to whom we wish to express our gratitude.

INTRODUCTION

FACTS FROM RELATED MATHEMATICAL DISCIPLINES. NOTATION. TERMINOLOGY

In the text we make use of a series of concepts and assertions from the theory of point sets. The terminology and notation that we employ correspond in general to those used in the book by N. Bourbaki, *General topology. Basic structures*, Fizmatgiz, Moscow, 1958 (*Éléments de mathématique*. Part I. *Les structures fondamentales de l'analyse*. Livre III. *Topologie générale*, Actual. Sci. Ind., nos. 858, 916, 1029, 1045, 1084, 1196. Hermann & Cie., Paris, 1940–1953). We will point out our changes from the terminology of Bourbaki whenever these changes might lead to confusion. The boundary of a set A will be denoted by ∂A. The symbol $\{\cdots\}$ will denote the set whose elements satisfy the conditions written in the brackets.

1. **Topological space.** We shall constantly in the sequel use the concept of a topological space. A set X is called a topological space if there are specified in X certain subsets, called open sets, for which the following axiom holds.

I. *The union of an arbitrary set of open sets is open and the intersection of a finite set of open sets is open.*

For convenience, we postulate that the entire set X and the void set are open sets.

This collection of open sets defines a *topological structure* or, briefly, a topology of the space X. We shall also use the concept of the induced topology in a set $X' \subset X$. In this topology, open subsets of the set X' are defined as the intersections of X' with open sets of the space X. As a result of introducing the induced topology, the set X' becomes a topological subspace of the space X.

An open set U that contains a point P of a topological space will be called

a neighborhood of the point P.

II. *A topological space X is called Hausdorff if every two distinct points have nonintersecting neighborhoods.*

We shall use the concept of a function, and in particular a continuous function, defined on a certain topological space X, and assuming complex (or in a special case, real) values. Such a function establishes a correspondence between the points of the space X and the space of complex numbers C or of real numbers R. A generalization of a (continuous) function is the notion of a (continuous) mapping r of the space X into the topological space X^*. In this case we write $r: X \to X^*$. A continuous mapping $r: X \to X^*$ is called *homeomorphic* if the inverse mapping $r^{-1}: X^* \to X$ is also continuous. It is called *proper* if the inverse image of every compact set under this mapping is compact. It is called *nowhere exceptional* if the set of inverse images of every point under this mapping is discrete.

We shall also use the concept of a connected and a locally connected topological space and the concept of a connected component of a topological space. In considering locally connected spaces, we usually mean by a neighborhood of a point P a connected open set (region) containing this point. A continuous mapping $\xi: T \to X$ of the closed interval $T = \{0 \leq t \leq 1\}$ into a topological space X, or simply a continuous function $x(t)$ defined on this closed interval, is called a *path* or a *line element* connecting the initial point $x(0) \in X$ with the terminal point $x(1) \in X$. The space X is called linearly connected if every pair of points $x_1, x_2 \in X$ are connected in X by a certain path $\{x = x(t)\}$. The space X is called locally linearly connected if, for every point $x \in X$ and every neighborhood U_x of the point x, there exists a linearly connected neighborhood $V_x \subset U_x$ of the point x. One can show that a linearly connected space is always connected, and that a connected locally linearly connected space is always linearly connected. A topological space X is called simply connected if, for all points $a, b \in X$ and all paths ξ and ξ' connecting these points, there exist paths ξ_τ, where $0 \leq \tau \leq 1$, depending continuously on τ, connecting the points a and b, and such that $\xi_0 = \xi$, $\xi_1 = \xi'$.

One says that a nowhere dense subset N of a topological space X does not split this space X at a point $a \in X$ if every connected neighborhood U_a of the point a contains a neighborhood V_a of a such that the set $V_a \setminus N$ is open and connected. The set N splits the space X nowhere if it splits X at no

point $a \in X$.

If the set $N \subset X$ splits the space X nowhere, then every set $N' \subset N$ that is closed in the set N (with respect to the induced topology) splits the space X nowhere.

To close the present section, we point out conditions for extensibility of continuous mappings of topological spaces, which will be useful in the sequel:

1) Let X^* be an everywhere dense subset of the topological space X; let $r\colon X^* \to Y$ be a continuous mapping of X^* into the regular Hausdorff space Y. If $\lim r x^*$, for $x^* \in X^*$, exists whenever the point x^*, remaining in X^*, converges to a point $x \in X$, then the mapping r can be extended, and in fact uniquely, to a continuous mapping on the entire space X.

2) Let X be a locally connected topological space, let $N \subset X$ be a certain set that splits this space nowhere, and let $r\colon (X \setminus N) \to Y$ be a continuous mapping of the set $X \setminus N$ into a locally compact topological space Y. This mapping can be continuously extended, and in fact uniquely, to a mapping $\tilde{r}\colon X \to Y$, if there exists a proper nowhere exceptional mapping $\phi\colon Y \to Z$, where Z is a certain locally compact topological space for which the mapping $\phi \circ r\colon (X \setminus N) \to Z$ can be continuously extended over the entire space X. 1)

Here, as usual, the symbol $\phi \circ r$ denotes the superposition of the mappings r and ϕ.

2. **Manifolds.** A Hausdorff space X is called a manifold with degree of smoothness l (and we write that the manifold $X \in \mathcal{C}^l$) if, in addition to axioms I and II, X satisfies the following axiom, which defines the character of its topological structure.

III. *For the space X there exists a complete family (atlas) $\mathfrak{X}$ of charts (U_j, ψ_j) which are consistent with one another and have degree of smoothness l. Here $j \in J$, where J is a certain set of indices; U_j is an open subset, an "element" of the space X, where $\bigcup_{j \in J} U_j = X$; ψ_j is a homeomorphic mapping of the element U_j onto the ball*

$$\sum_{k=1}^{p_j} (t_k^j)^2 < 1$$

1) The proof of this assertion and of a much stronger assertion can be found in the work of Stein [2].

in the space of auxiliary variables, the "uniformizing" parameters $t_1^j, \cdots, t_{p_j}^j$. *These parameters are also sometimes called local coordinates of the points of the element* U_j. *If the intersection* $U_i \cap U_j \neq \emptyset$, *then the homeomorphism* $\psi_j(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ *is smooth of degree* l, *that is, it is given by means of functions* $t_k = f_k(t_1^i, \cdots, t_{p_i}^i)$, $k = 1, \cdots, p_j$, *having continuous partial derivatives of all orders up to and including* l (*this is what we mean by consistency of charts of the atlas* $\mathfrak{X}$). *It is assumed that every chart* (V, ϕ), *consistent with degree* l *with all charts of the atlas* $\mathfrak{X}$, *actually belongs to* X (*this is what we mean by saying that the atlas is complete*).

The degree of smoothness l of the manifold X can also be infinite (then we write: $X \in \mathcal{C}^\infty$). If all of the functions $f_k(t_1^i, \cdots, t_{p_i}^i)$ are analytic functions of the real variables $t_1^i, \cdots, t_{p_i}^i$, the manifold X is called real analytic (then we write: $X \in \mathcal{C}^\infty$). If the degree of smoothness $l = 1$, the manifold X is called simply smooth. If the manifold X can be dissected into a finite set of l-smooth manifolds, then it is called piecewise l-smooth. If the degree of smoothness $l = 0$, then we shall naturally not speak of smoothness of the manifold at all.

In a number of cases, it is convenient to take for the region $\psi_j U_j$ not the ball $\Sigma_{k=1}^{p_j} (t_k^j)^2 < 1$, but another region: an open proper simplex, an open curvilinear simplex (defined in section 3 of this introduction), or a polycylinder (defined in §2, Chapter I).

It follows from axiom III that every point x of the manifold X has a neighborhood that is homeomorphic with a certain euclidean space. A point of an arbitrary topological space having this property is said to be *uniformized.* Thus a manifold consists entirely of uniformized points.

Suppose that every point of a manifold $x \in U_j$ has a neighborhood homeomorphic to euclidean space of dimension p_j. The number p_j is the topological dimension of the element U_j, and it is also called the topological dimension of the chart (U_j, ψ_j). It follows from axiom III that if $U_i \cap U_j \neq \emptyset$, then $p_i = p_j$. Thus all charts belonging to one and the same connected component X_κ of the manifold X have one and the same dimension $\mathrm{Dim}(X_\kappa)$. If we write the collection of connected components of the manifold X as $(X_\kappa, \kappa \in K)$, where K is a certain set of indices, then the quantity $\mathrm{Dim}(X) = \sup_{\kappa \in K} \mathrm{Dim}(X_\kappa)$ is called the topological dimension of the manifold X; this dimension can be infinitely

large. If all of the connected components of the manifold X have one and the same dimension p, then it is called homogeneous or purely dimensional. In this case, we write $X = X_p$. If all of the topological dimensions $\mathrm{Dim}(X_\kappa)$ are even, then the numbers $\frac{1}{2}\mathrm{Dim}(X_\kappa) = \dim(X_\kappa)$, $\frac{1}{2}\mathrm{Dim}(X) = \dim(X)$, are called the complex dimensions of the corresponding sets X_κ and X. If the manifold X has a pure even topological dimension $p = 2d$, then we will write $X = X^d$.

If $Y_q \subset X_p$, where Y_q is a certain q-dimensional submanifold of the manifold X_p (for the definition of submanifold in the case that we need, see section 3 of this introduction), then the number $p - q$ is called the topological codimension of the submanifold Y_q in the manifold X_p. If $p = 2d$, $q = 2e$, then the number $d - e$ is the complex codimension of the manifold Y^e in the manifold X^d.

In the sequel, we shall consider topological and complex dimensions and codimensions not only for manifolds, but also for topological spaces of a more general form.

Below we will let R_N denote N-dimensional euclidean space. If $N = 2n$ and the quantities x_k, y_k $(k = 1, \cdots, n)$ are cartesian coordinates of points of the space R_{2n}, then it is called the space of complex variables $z_k = x_k + iy_k$ $(k = 1, \cdots, n)$ and is denoted by the symbol C^n. Furthermore, we write $R_1 = R$, $C^1 = C$.

In the sequel, when we call a certain set a region (neighborhood of a point), we always assume that it is a region (neighborhood of a point) in a space of the largest dimension among all of the dimensions under consideration. If a certain set is a region (neighborhood of a point) in a subspace, then this fact will always be pointed out specifically.

3. **Submanifold. Surface.** A mapping $\mu : Y_k \longrightarrow X_N$ of an l_1-smooth manifold Y_k into an l_2-smooth manifold X_N $(k \leq N)$ is called l-smooth, or is said to belong to the class C^l, if for every point $y^{(0)} \in Y_k$, there exists a chart (V, ϕ), $y^{(0)} \in V$, of the manifold Y_k and a chart (U, ψ), $\mu y^{(0)} = x^{(0)} \in U$, of the manifold X_N, such that the local coordinates $x_1, \cdots, x_N$ of the point $\mu y = x \in U$, when considered as functions of the local coordinates $y_1, \cdots, y_k$ of the point $y \in V$, that is,

$$x_p = x_p(y_1, \cdots, y_k \,|\, y^{(0)}), \quad p = 1, \cdots, N, \tag{0.1$_1$}$$

admit continuous derivatives of the first l orders. Here we set $l \leq \min(l_1, l_2)$. For $l = \infty$ the mapping μ is called infinitely smooth or is said to belong to the

class $\mathcal{C}^\infty$, and the class $\mathcal{C}^\omega$ is defined analogously. These definitions apply in particular to functions defined on the manifold Y.

If the rank of the matrix consisting of the derivatives of the functions (0.1_1) always has the largest possible value k, then the mapping μ is called a *locally regular imbedding* of the manifold Y_k in the manifold X_N. This imbedding is called regular if in addition the mapping μ: $Y_k \rightarrow \mu Y_k \subset X_N$ is one-to-one.

It is easy to see that all these properties of the mapping μ are independent of the choice of the system of local coordinates.

In view of known theorems on implicit functions, if an imbedding is locally regular, the equalities (0.1_1) can be replaced by the relations

$$s_p(x_1, \cdots, x_N, y_1, \cdots, y_k \,|\, y^{(0)}) = 0, \quad p = 1, \cdots, N. \qquad (0.1_2)$$

Here the functions s_p satisfy the corresponding conditions of smoothness, and the matrices formed from their partial derivatives of the first order have the appropriate rank.

We also remark that in the case under consideration (i.e., of a locally regular imbedding μ: $Y_k \rightarrow X_N$), for every point $y^{(0)} \in Y_k$, one can find a chart (V, ϕ), $y^{(0)} \in V$, and k indices, $1 \leq i_1 < \cdots < i_k \leq N$, such that the functions $x_{i_q}(y \,|\, y^{(0)})$ from (0.1_1) play the role of local coordinates on the element V. Then the equations (0.1_1) and (0.1_2) assume, respectively, the following forms:

$$x_p = x_p(x_{i_1}, \cdots, x_{i_k} \,|\, y^{(0)}), \qquad (0.1_3)$$

$$s_q(x_1, \cdots, x_N \,|\, y^{(0)}) = 0, \quad q = 1, \cdots, N - k. \qquad (0.1_4)$$

Here the index p runs from 1 to N, omitting the values $i_1, \cdots, i_k$, and thus assumes $N - k$ values.

If $Y_k \subset X_N$, we will speak, respectively, of a locally regularly or regular imbedded *submanifold* Y_k of the manifold X_N.

In the sequel we shall most frequently have to do with an imbedding of a certain manifold Y_k in the space R_N of variables $x_1, \cdots, x_N$. If also we have $Y_k \subset R_N$, then an l-smooth locally regular imbedding μ: $Y_k \rightarrow R_N$ leads to the description of each element V of the manifold Y_k by means of equations

$$(\phi) \qquad x_p = x_p(t_1, \cdots, t_k \,|\, y^{(0)}), \quad p = 1, \cdots, N,$$

$$(t_1, \cdots, t_k) \in T. \qquad (0.1_5)$$

Here $\phi V = W$ is a ball or an open simplex in the space of local coordinates $t_1, \cdots, t_k$ on the element V. We shall also usually assume that the functions $x_p(t_1, \cdots, t_k)$ belong to the class $\mathcal{C}^l$, and that the rank of the matrix $\| \partial x_p / \partial t_s \| = k$ in the closed region $\overline{W}$. Then the boundary ∂V of such an element V can be described by the equations (0.1_5), where $(t_1, \cdots, t_k) \in \partial W$. Here ∂W is the boundary of the region W in the space of local coordinates $t_1, \cdots, t_k$.

In this case, the element V is called an l-smooth k-dimensional *surface element* or a *curvilinear simplex* in the space R_N (if we use the latter term, then for W we take an open proper simplex). If the manifold Y_k is connected and admits a covering by a countable set $\{V\}$ of such surface elements, it is called an l-smooth k-dimensional surface in the space R_N. For $k = N - 1$, it is called a hypersurface.

Furthermore, we shall also use the concepts of a piecewise smooth surface, a closed surface, surface with boundary, and self-intersecting surface.

4. Orientation of a manifold. Let X_N be an l-smooth (where $l \geq 1$) or a real analytic N-dimensional, connected manifold. For each point $x \in X_N$ we consider a certain nonempty collection P_x of its charts (U, ψ) for which $x = \psi^{-1}(0) \in U$. Here the point 0 is the origin of the space R_N of corresponding local coordinates $t_1, \cdots, t_N$.

We consider the case when there exists a set $\Pi = \{P_x, x \in X_N\}$ of such collections of charts, possessing the following property: for every pair of charts $(U, \psi) \in P_x$, $(U^*, \psi^*) \in P_{x^*}$, where $x, x^* \in X_N$, if $U \cap U^* \neq \emptyset$ and $t = (t_1, \cdots, t_N) \in (U \cap U^*)$, then the Jacobian

$$\frac{\partial t^*}{\partial t} = \frac{\partial(t_1^*, \cdots, t_N^*)}{\partial(t_1, \cdots, t_N)} > 0.$$

Here $t_1^*, \cdots, t_N^*$ are local coordinates in the space $R_N^* \supset \psi^*(U^*)$. Then one says that the manifold X_N is oriented.

Suppose that $\Pi^{(+)} = \{P_N^{(+)}, x \in X_N\}$ is one of such sets of collections of charts of the manifold X_N. With every chart $(U, \psi^{(+)}) \in P_x^{(+)}$ we associate the chart $(U, \psi^{(-)})$, where $\psi^{(-)} = \psi^{(+)} \circ E$. Here E is the mapping of the space R_N onto itself defined by the condition: $(t_1, t_2, \cdots, t_N) \longrightarrow (t_2, t_1, \cdots, t_N)$. We form the collection $P_x^{(-)}$ of charts $(U, \psi^{(-)})$ for each point $x \in X_N$ and then the set of these collections $\Pi^{(-)} = \{P_x^{(-)}, x \in X_N\}$. The set $\Pi^{(-)}$ also satisfies the requirement stated above. We associate with the set $\Pi^{(+)}$ the number

$\epsilon(\Pi^{(+)}) = +1$, and with the set $\Pi^{(-)}$ the number $\epsilon(\Pi^{(-)}) = -1$. These numbers will be called orientations of the manifold X_N defined by the set $\Pi^{(+)}$ and $\Pi^{(-)}$, and the manifold X_N, supplied in this way with an orientation, will be said to be oriented.[1] We denote the positively oriented manifold X_N by $X_N^{(+)}$, and the negatively oriented one by $X_N^{(-)}$. We shall also write $X_N^{(-)} = -(X_N^{(+)})$. When m is an integer, we usually denote by $mX_N^{(+)}$ a collection of $|m|$ manifolds $X_N^{(+)}$ oriented in conformity with the sign of the number m.

It is clear that the choice of the set $\Pi^{(+)}$, which we used to define the positive orientation of the manifold X_N, is arbitrary. We can use an arbitrary set of collections of charts that satisfies the condition indicated above. It is also obvious that the Jacobian of the transformation from local coordinates on charts belonging to the set $\Pi^{(+)}$ to local coordinates on charts belonging to the set $\Pi^{(-)}$ is always negative. Hence $\epsilon(\Pi^{(-)}) = (J/|J|)\,\epsilon(\Pi^{(+)})$.

We consider two sets $\Pi^{(k)} = \{P_x^{(k)},\ x \in X_N\}$, $k = 1, 2$, defining certain orientations of the manifold X_N. If we have $P_x^{(1)} \subset P_x^{(2)}$ for all $x \in X_N$, then the set $\Pi^{(2)}$ is called an extension of the set $\Pi^{(1)}$. We form the set $\Pi = \{\bigcup_{k\in K} P_x^{(k)},\ x \in X_N\}$, where K is a set of indices corresponding to all possible extensions of a certain original set $\Pi^{(0)}$. The set Π of collections of charts is called maximal. All orientations of the manifold X_N, defined by means of sets $\Pi^{(k)} \subset \Pi$, $k \in K$, are taken to be identical. Ordinarily we define the orientation of the manifold X_N by means of a maximal set Π.

We say of an orientation $\epsilon(\Pi) = \pm 1$ that it is defined by means of systems of local coordinates given on the charts $(U, \psi) \in \Pi$. We also call it the orientation of these systems of coordinates.

In the sequel we shall use the operation of triangulation of a manifold X_N, which is intimately connected with the concept of its orientation.

5. **Orientation of a mapping.** We consider a mapping μ of class $\mathcal{C}^l$ (where $l \geq 1$) of a connected manifold Y into a manifold X. The mapping μ is called *oriented* if one can choose orientations $\epsilon'(U_k)$ and $\epsilon''(V_k)$ of all pairs of oriented regions $U_k \subset X$ and $V_k \subset Y$, where $\mu V_k \subset U_k$ and $k \in K$ (here K is a

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1) We remark that if the region $\psi(U)$ is a simplex in the space of local coordinates, then the transformation E corresponds to a permutation of the vertices of this simplex. Thus there is established a correspondence between the orientation of a curvilinear simplex U and the order in which its vertices are given. The transferral from the positive to the negative orientation of the simplex U corresponds to an odd permutation of its vertices.

certain set of indices) such that if $U_{k_1} \cap U_{k_2} \neq \emptyset$, then $\epsilon'(U_{k_1})\,\epsilon'(U_{k_2}) = \epsilon''(V_{k_1})\,\epsilon''(V_{k_2})$.

It follows from the last equality that the quantity $\epsilon = \epsilon'\epsilon'' = \pm 1$ is constant for all $k \in K$. We complete the definition of the mapping $\mu: Y \to X$: we agree that it juxtaposes the orientations ϵ' and ϵ'' of the regions U_k and V_k, and we will call the quantity $\epsilon = \epsilon'\epsilon''$ the orientation of the mapping μ.

If the manifolds X and Y are oriented, then the mapping $\mu: Y \to X$ is also oriented. If the set μY has an oriented neighborhood in the manifold X (in particular, if the manifold X is oriented), then the orientation of the manifold Y and the orientation of the mapping μ are identical. Knowing the orientation of the mapping μ and of one of the manifolds X and Y, it is easy to find the orientation of the other manifold.

If the mapping being studied, $\mu: Y_N \to X_N$, is a homeomorphism, and if $t_k(y)$ $(k = 1, \cdots, n)$ are local coordinates on a certain chart (V, ϕ), where $V \subset Y$ and $y \in V$, then the functions $t_k(\mu^{-1} X)$ form a system of local coordinates on the corresponding chart (U, ψ), where $U \subset X$, and $X = \mu y \in U$. Here $\mu^{-1}: X \to Y$ is the mapping inverse to μ. If ϵ_1 and ϵ_2 are the orientations of these systems of local coordinates, then the quantity $\epsilon = \epsilon_1 \epsilon_2$ is called the *canonical orientation* of the homeomorphism μ (corresponding to the local coordinates $t_1, \cdots, t_N$).

6. **Chain. Homology group.** We now define an l-smooth (where $l \geq 0$) oriented element of an N-dimensional chain v in a manifold X. This is a pair $(\bar{w}, \mu)$, where w is an oriented open simplex from the space R_N of variables $x_1, \cdots, x_N$, and $\mu: \bar{w} \to X$ is an l-smooth oriented mapping. We assume that the orientation of the simplex w coincides with the orientation of the entire space R_N and is defined by the system of coordinates $x_1, \cdots, x_N$. By a (finite) chain $\mathfrak{S}_N$ in the manifold X we ordinarily understand a (finite) linear combination $\sum c_k v_k$ of elements of a chain $v_k = (w_k, \mu_k)$ with integer coefficients c_k. In the integral calculus we will sometimes meet chains with arbitrary complex coefficients. The meaning attached to such a chain will be pointed out below. The chain $\mathfrak{S}_N$ is not changed if we break up each of the simplices w_k into the sum of a number of correspondingly oriented simplices. From this follows the possibility of defining one and the same chain $\mathfrak{S}_N$ with the aid of different systems of simplices w_k.

Suppose that $(\partial w)_1$ is the $(N-1)$-dimensional boundary of a simplex w

which is positioned in such a way that the unit vector of the axis x_1 with initial point at a certain point $x \in (\partial w)_1$ lies entirely outside of the closed simplex w. Then the variables $x_2, \cdots, x_N$ serve as local coordinates on the boundary $(\partial w)_1$ and define on it (when considered in the order indicated above) an *induced* or *coherent* (with respect to the orientation of the open simplex w) orientation. The coherent orientation of the remaining boundaries of the simplex w is defined analogously. The collection of all of them comprises the entire coherently oriented boundary ∂w of the simplex w, and the pair $\partial v = (\partial w, \mu)$ defines the oriented boundary of the N-dimensional element of the chain v. This boundary is an $(N-1)$-dimensional chain in the manifold X.

The boundary of the chain $\Sigma c_k v_k$ is defined by the formula

$$\partial(\Sigma c_k v_k) = \Sigma c_k (\partial v_k). \tag{0.2}$$

A chain whose boundary is equal to zero is called a cycle. One can show that the boundary of an arbitrary chain is always a cycle. Two cycles $\mathfrak{S}_1$ and $\mathfrak{S}_2$ for which the chain $\mathfrak{S}_1 - \mathfrak{S}_2 = \partial \mathfrak{S}$, where $\mathfrak{S}$ is a certain chain, are called homologous to each other: in this case we write $\mathfrak{S}_1 \sim \mathfrak{S}_2$.

Thus the entire collection of cycles on a manifold X is divided into homologous classes. Each of these consists of cycles that are homologous to some fixed cycle. Such a class is denoted by the symbol $h(X)$. If the cycle $v' \in h_1(X)$, and the cycle $v'' \in h_2(X)$, then the class $h(X)$ containing the cycle $v' + v''$ is regarded as the sum of the classes $h_1(X)$ and $h_2(X)$.

Under the operation of addition, the set of homology classes becomes an abelian homology group on the manifold X. This group is denoted by the symbol $H(X)$.

We will use in the sequel certain properties of the homology group, and also some related concepts: groups of compact homologies, groups of compact homologies of the manifold X relative to its submanifold X_0. One can study these, for example, in the book of G. de Rham, *Variétés différentiables*, Actualités Sci. Ind. No. 1222, Hermann, Paris, 1955.

7. **Exterior forms.** Let X_N be an l-smooth $(l \geq 1)$ manifold, and let X be a subset of X_N. An even p-covector, or in other words a skew-symmetric covariant tensor of rank p, is defined on the set X by specifying its components $a_{i_1 \ldots i_p}$, where $1 \leq i_k \leq N$, and $k = 1, \cdots, p$, as (generally speaking) complex functions of the point $t(t_1, \cdots, t_N) \in \psi(U \cap X)$ on every chart (U, ψ) of the

given manifold for which $U \cap X \neq \emptyset$. These components are skew-symmetric in the indices $i_1, \cdots, i_p$; the components $a^*_{j_1 \cdots j_p}$ of the same p-covector in any other chart (U^*, ψ^*), for which $U \cap U^* \cap X \neq \emptyset$, are functions of the point $t^*(t^*_1, \cdots, t^*_N) \in \psi^*(U^* \cap X)$ and are connected with the components $a_{i_1 \cdots i_p}$ by the relations

$$a^*_{j_1 \cdots j_p} = \sum_p^N a_{i_1 \cdots i_p} \frac{\partial t_{i_1}}{\partial t^*_{j_1}} \cdots \frac{\partial t_{i_p}}{\partial t^*_{j_p}} = \sum_p^{N}{}' a_{i_1 \cdots i_p} \frac{\partial(t_{i_1} \cdots t_{i_p})}{\partial(t^*_{j_1} \cdots t^*_{j_p})}. \tag{0.3$_1$}$$

Here and in the sequel the symbol Σ_p^N denotes summation over all values of the indices $i_1, \cdots, i_p = 1, \cdots, N$; the symbol Σ'^N_p denotes summation over all values of these indices satisfying the conditions $1 \leq i_1 < i_2 < \cdots < i_p \leq N$.

The definition of an odd p-covector differs from the definition of an even p-covector only in that the formulas of transformation to its components in the new local coordinates have the form

$$a^*_{j_1 \cdots j_p} = \frac{J}{|J|} \sum_p^{N}{}' a_{i_1 \cdots i_p} \frac{\partial(t_{i_1}, \cdots, t_{i_p})}{\partial(t^*_{j_1}, \cdots, t^*_{j_p})}. \tag{0.3$_2$}$$

Here we have

$$J = \frac{\partial(t_1, \cdots, t_N)}{\partial(t^*_1, \cdots, t^*_N)}.$$

In particular, for $p = N$, we obtain

$$a^*_{1 \cdots N} = |J| \, a_{1 \cdots N}. \tag{0.3$_3$}$$

Setting $p = 0$, we obtain 0-covectors or scalars of even and odd kind. It is evident that the orientation of a manifold is a scalar of odd kind. A manifold is oriented if such a continuous scalar ϵ (where $\epsilon^2 = 1$) can be defined on it.

With every even or odd p-covector $a_{i_1 \cdots i_p}$ there is directly connected an even or odd exterior differential form α of degree p. In local coordinates $t_1, \cdots, t_N$, it is written in the form

$$\alpha = \sum_p^{N}{}' a_{i_1 \cdots i_p} \, dt_{i_1} \wedge \cdots \wedge dt_{i_p} = \frac{1}{p!} \sum_p^N a_{i_1 \cdots i_p} \, dt_{i_1} \wedge \cdots \wedge dt_{i_p}. \tag{0.4}$$

The operation of exterior multiplication, denoted by the symbol $\wedge$, obeys the laws of associativity and distributivity, and the following laws of

pseudocommutativity:

$$dt_i \wedge dt_j = - dt_j \wedge dt_i; \quad dt_i \wedge dt_i = 0,$$
$$a \wedge dt_i = dt_i \wedge a = adt_i; \quad dt_i \wedge adt_j = adt_j \wedge dt_j; \tag{0.5}$$
$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

Here a is a certain scalar and β is an exterior differential form of degree q. A certain covector and the corresponding form are said to belong to the class $\mathcal{C}^l$ $(0 \leq l \leq \infty)$ if the components of this covector have continuous partial derivatives of the first l orders in local coordinates on the manifold X. We then assume that the degree of smoothness of the manifold X is not less than the number l. For $l = 0$, the form is simply called continuous and we make no mention of its smoothness. An odd form α defined on an oriented manifold can always be represented in the form $\alpha = \epsilon\alpha_1$, where α_1 is an even form and ϵ is the orientation of the manifold.

8. The differential of the form α of degree p and class $\mathcal{C}^1$ in local coordinates is defined by the equality

$$d\alpha = \sum_p^{N}{}' \, da_{i_1 \ldots i_p} \wedge dt_{i_1} \wedge \cdots \wedge dt_{i_p}. \tag{0.6}$$

It is easy to see that $d\alpha$ is an exterior differential form of degree $p + 1$. The following equalities hold:

$$d(\alpha_1 + \alpha_2) = d\alpha_1 + d\alpha_2;$$
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta, \quad d^2\alpha = 0. \tag{0.7}$$

The last equality holds for forms $\alpha \in \mathcal{C}^l$ $(l \geq 2)$. A form α for which $d\alpha = 0$ is said to be closed. It follows from the last equality (0.7) that a form $d\alpha$ is always closed.

Two closed forms $\alpha, \beta \in \mathcal{C}^\infty$ on the manifold X are said to be cohomologous with each other if $\alpha - \beta = d\gamma$, where the form $\gamma \in \mathcal{C}^\infty$ on the manifold X. In this case, we write $\alpha \sim \beta$. These forms α and β are called compactly cohomologous with each other, if the form γ has a compact carrier (for the concept of a compact carrier, see the following section).

Thus the entire collection of closed forms $\alpha \in \mathcal{C}^\infty$ on the manifold X is divided into a class of (compact) cohomologies. Each of them consists of forms

that are (compactly) cohomologous to a certain closed form $\alpha_0 \in \mathcal{C}^\infty$. Such a class of cohomologies is denoted by the symbol $h^*(X)$, and a class of compact cohomologies by the symbol $h_c^*(X)$.

If the form $\alpha_1 \in h_1^*(X)$ and the form $\alpha_2 \in h_2^*(X)$, then the class $h^*(X)$ containing the form $\alpha_1 + \alpha_2$ is considered as the sum of the classes $h_1^*(X)$ and $h_2^*(X)$.

It is easy to see that if $\alpha_1 \sim \alpha_2$, $\beta_1 \sim \beta_2$, and $\alpha_1 \wedge \beta_1 \sim \alpha_2 \wedge \beta_2$, then we have $\alpha_2 \wedge \beta_2 - \alpha_1 \wedge \beta_1 = d(\alpha \wedge \beta_2 + (-1)^p \alpha_1 \wedge \beta)$, where $\alpha_2 - \alpha_1 = d\alpha$, and $\beta_2 - \beta_1 = d\beta$. Here p is the degree of the form α_1.

This fact permits us to define the product of classes of cohomologies $h_1^*(X)$ and $h_2^*(X)$ as the class $h^*(X)$ containing the form $\alpha_1 \wedge \alpha_2$, if $\alpha_1 \in h_1^*(X)$ and $\alpha_2 \in h_2^*(X)$.

With the operations of addition and multiplication, the collection of classes of cohomology $h^*(X)$ becomes the ring of cohomology of closed forms $\alpha \in \mathcal{C}^\infty$ on the manifold X. This ring is denoted by the symbol $H^*(X)$.

The ring of compact cohomology of these forms, denoted by the symbol $H_c^*(X)$, is defined in an analogous fashion.

We will consider closed forms $\alpha \in \mathcal{C}^\infty$ on the manifold X, vanishing on a submanifold $X_0 \subset X$. Proceeding as above, we construct the classes of cohomology $h^*(X, X_0)$, the classes of compact cohomologies $h_c^*(X, X_0)$ relative to the submanifold X_0, and from these the rings of relative cohomologies $H^*(X, X_0)$ and $H_c^*(X, X_0)$.

9. **Supplementary facts from the theory of exterior forms.** Consider a mapping $\mu: Y \to X$, where the manifolds X and Y, as well as the mapping μ, are supposed to be smooth. To every form α given on the manifold X, there corresponds its inverse image $\alpha \circ \mu$ on the manifold Y. If an even form α is given in local coordinates $t_1, \cdots, t_N$ by the equality (0.4), then in the corresponding local coordinates on the manifold Y we have

$$\alpha \circ \mu = \sum_p^N{}' (a_{i_1 \ldots i_p} \circ \mu)\, d(t_{i_1} \circ \mu) \wedge \cdots \wedge d(t_{i_p} \circ \mu). \tag{0.8}$$

Here $t_k \circ \mu$ is the expression of the variable t_k by local coordinates on the manifold Y.

If the manifolds X and Y are oriented, and ϵ' and ϵ'' are their orientations (we limit ourselves to this case), the inverse image of an odd form α is defined

by the equality

$$\alpha \circ \mu = \epsilon[(\epsilon' \alpha) \circ \mu]. \tag{0.9}$$

The carrier of the continuous form α on the manifold X (in particular, of a continuous function on the manifold X) is defined as the smallest closed set $\mathfrak{X} \subset X$, outside of which all of its coefficients are equal to zero. The carrier of the form α is called bounded, if there exists a compact submanifold of the manifold X that contains it. If $X = R_N$, then a form (in particular, a function) α with a bounded carrier is called finite.

In the following exposition we shall also use certain other facts from the theory of exterior differential forms. These facts can be found, for example, in the above-mentioned book of G. de Rham, *Differentiable manifolds*.

10. A bundle* over a topological space. In the following exposition we shall use a series of algebraic concepts. We shall adhere to the definitions and notations given in the book *Modern algebra* of van der Waerden, Parts 1 and 2.

The concept of a bundle over a topological space will also play an important role. A bundle of abelian groups $\{F_x, x \in X\}$ over a topological space X consists of: a) a certain correspondence $x \to F_x$, assigning the abelian group F_x to an arbitrary point $x \in X$ (we write the group additively); b) a certain topology in the union F of the sets F_x (in other words, there is defined in the set F a system of open sets satisfying the axioms of a topological space). For an arbitrary element $f \in F_x$, we set $\pi(f) = x$. The mapping $\pi: F \to X$ is called the *projection* of the bundle space F onto the space X. The subset of the product $F \times F$, formed of those pairs (f, g) (where $f \in F$, $g \in F$) such that $\pi(f) = \pi(g)$, is further denoted by the symbol $F + F$.

The correspondence defined in a), and the topology referred to in b) must satisfy the following axioms:

I. The mapping π is a local homeomorphism (that is, every element $f \in F$ has a neighborhood V which is mapped homeomorphically by the projection π onto a certain neighborhood U of the point $\pi(f)$).

II. The correspondence $f \to -f$ is a continuous mapping of the space F onto itself; the correspondence $(f, g) \to f + g$ is a continuous mapping of the set $F \times F$ into the space F.

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*Translators' note: The word pučok has everywhere been translated by "bundle"; in some passages it could have been translated by "faisceau" or "sheaf".

Let F be the space of the bundle $\{F_x, x \in X\}$ and $\zeta: V \to F$ a continuous mapping of a certain subset V of the space X onto the bundle space F, where the composition $\zeta \circ \pi$ is the identity mapping $V \to V$. Then the set $\zeta(V) \subset F$ is called a *section* of the bundle $\{F_x, x \in X\}$ over the set $V \subset X$.

Let $\{F_x, x \in X\}$ be an arbitrary bundle over the space X, let F be the space of this bundle, and let G be an open subset of the set F, such that for every point $x \in X$ the intersection $G \cap F_x = G_x$ is a subgroup of the group F_x. Then the collection $\{G_x, x \in X\}$ is a bundle with the relative topology of G as a subspace of F (that is, open sets of the space G are defined as intersections with G of open sets of the space F). The bundle $\{G_x, x \in X\}$ is called a subbundle of the bundle $\{F_x, x \in X\}$.

The concept of a bundle is applicable not only to abelian groups; analogous definitions can be formulated also for other algebraic objects.

In the following we shall use the concepts of bundles of commutative rings, bundles of commutative rings with units, and bundles of ideals. We shall also consider *bundles of sets*. In this last case, we assign no algebraic structure to the corresponding sets F_x; they need only satisfy Axiom I in the definition of a bundle.

The *bundles of rings* with units $\{\mathfrak{O}_x, x \in X\}$ which are considered below are collections of commutative rings with units $\mathfrak{O}_x$, which are in correspondence with the points x of a certain topological space X. Such a bundle is defined as the bundle of abelian groups $\mathfrak{O}_x$ (all the rings $\mathfrak{O}_x$ are abelian groups) over the space X. The correspondence $(f, g) \to fg$ (where $f \in \mathfrak{O}$, $g \in \mathfrak{O}$, and $\mathfrak{O}$ is the space of the bundle) is also to be a continuous mapping of the set $\mathfrak{O} + \mathfrak{O}$ into the space $\mathfrak{O}$. It is further assumed that the unit of the ring changes continuously with the point $x \in X$.

An important example of a bundle of rings of integrity is furnished by the bundle $\mathfrak{G}(D) = \{\mathfrak{G}_x, x \in D\}$ of rings of germs of continuous functions $\mathfrak{G}_x$ over a certain region $D \subset R_n$. A germ $\mathfrak{G}_{x_0}$ of a continuous function at the point $x_0 \in D$, represented by a certain function $g(x)$ that is continuous at the point x_0, is defined as the set of all functions continuous at the point x_0 and agreeing with the function $g(x)$ is a certain neighborhood of the point x_0. These neighborhoods may differ for different functions belonging to the germ $\mathfrak{G}_{x_0}$. It is plain that such a germ can be represented by means of any function that belongs to it.

The arithmetic operations on germs $\mathfrak{G}_x$ are defined by means of the operations on continuous functions belonging to them. It is clear that the set of such germs, defined at a certain point $x \in D$, becomes a ring of integrity with unit; and this ring we will denote by $\mathfrak{G}_x$.

We define a topology in the set $\mathfrak{G}$ of germs $\mathfrak{G}_x$ as follows. Let $U_{x_0} \subset D$ be a neighborhood of the point $x_0 \in D$. A neighborhood of the germ $\mathfrak{G}_{x_0}$ is defined as the set V of germs represented at the points $x \in U_{x_0}$ by a certain function $g(x)$ that is continuous throughout the neighborhood U_{x_0} and which represents the germ $\mathfrak{G}_{x_0}$ at the point x_0. Considering different continuous functions $g(x)$ belonging to the germ $\mathfrak{G}_{x_0}$, we obtain different neighborhoods of this germ. The sets V form a basis of the topology in the set $\mathfrak{G}$. As a result $\mathfrak{G}$ becomes a topological space. It is eaxy to verify that under our conditions the requirements of Axioms I and II in the definition of a bundle are satisfied.

We also remark that the topology defined in the space of the bundle $\mathfrak{G}$ is not, generally speaking, Hausdorff. For example, we consider continuous functions of one real variable x, and let D be the axis OX. Then the bundles of continuous functions represented at the point $x = 0$ by the functions $x + |x|$ and $-(x + |x|)$ do not have separated neighborhoods.

11. **Supplementary remarks.** We shall not use any special topics of the theory of analytic functions of one complex variable. Therefore I. I. Privalov's *Introduction to the theory of functions of a complex variable*, 9th ed., GITTL, Moscow, 1954 (Russian), contains everything that we shall need here. It goes without saying that these facts can be obtained also from other books devoted to this subject. In various places of our book, we draw on facts from other mathematical disciplines as well. The corresponding references in the text contain in such cases the necessary bibliographical directions.

CHAPTER I

FUNDAMENTAL PROPERTIES OF HOLOMORPHIC FUNCTIONS IN A SPACE OF n COMPLEX VARIABLES

§1. FUNCTIONS OF n COMPLEX VARIABLES, THEIR DIFFERENTIATION AND INTEGRATION. HOLOMORPHIC FUNCTIONAL ELEMENT

1. Continuous functions of n complex variables. [1] Let D be a subset of the space C^n_z of complex variables $z_k = x_k + iy_k$ $(k = 1, \cdots, n)$.

If to each point $z \in D$ there is put into correspondence one or several complex numbers f, then we will say that on the set D there is defined a *function* $f = f(z) = f(z_1, \cdots, z_n)$. Here $z_1, \cdots, z_n$ are the coordinates of the point z. If to each point $z \in D$ there corresponds one number f, then the function $f(z)$ is said to be single-valued. In what follows, unless otherwise specified, we will always deal with single-valued functions.

If the set $E \subset D$, the function $f_1(z)$ is defined on the set E and $f_1(z) = f(z)$ for $z \in E$, then the function $f_1(z)$ is said to be the *restriction* or *trace* of the function $f(z)$ on E; in this case we write $f_1 = f | E$. The designation "trace of a function" is usually employed when the set $E \subset C^n$ is a manifold (or space) of complex dimension $< n$. The function $f(z)$ is said to be *continuous* at the point $z \in D$, if to each number $\epsilon > 0$ there corresponds a number $\delta = \delta(\epsilon, z)$ such that if $\sqrt{|\Delta z_1|^2 + \cdots + |\Delta z_n|^2} < \delta$ and $(z_1 + \Delta z_1, \cdots, z_n + \Delta z_n) \in D$ the function $f(z_1 + \Delta z_1, \cdots, z_n + \Delta z_n)$ is defined and

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1) The definitions and theorems contained in this section are supposed to be known to the reader, and are presented here only for convenience of reference.

$$|f(z_1 + \Delta z_1, \cdots, z_n + \Delta z_n) - f(z_1, \cdots, z_n)| < \epsilon.$$

If for each number ϵ the quantity $\delta(\epsilon, z)$ can be chosen to be the same for all $z \in D$, i.e., if one can take $\delta = \delta(\epsilon)$, then the function $f(z)$ is *uniformly continuous* on the set D.

As usual, one establishes the usual theorems of the theory of continuous functions, due to Cantor and Weierstrass, formulated below for the case of a region.

THEOREM 1.1. *A function continuous at all the points of a bounded closed region is uniformly continuous in that region.*

THEOREM 1.2.[1)] *A function continuous at every point of a bounded closed region is bounded in that region, i.e., there exists a number $M > 0$ such that for all the points z of that region $|f(z)| < M$.*

THEOREM 1.3.[1)] *A real function continuous at every point of a bounded closed region D has in that region maximum and minimum values.*

In what follows we shall constantly use both ordinary and multiple numerical and functional series, and the concepts of absolute and uniform convergence for them. We suppose these general properties known to the reader. In particular we suppose it is known that the functional series $\sum_{k=1}^{\infty} f_k(z)$ whose terms are bounded at the points z of a region D of the space C^n *converges uniformly* in that region if to each number $\epsilon > 0$ and compact set M lying in the region D there corresponds a number $N = N(\epsilon, M)$ such that for $n > N$ and for all $z \in M$

$$\left| \sum_{k=1}^{n} f_k(z) - g(z) \right| < \epsilon.$$

In the same way we suppose the following theorems known:

THEOREM 1.4. *If the numerical series $\sum_{k=1}^{\infty} a_k$, where $a_k \geq 0$, converges and if the inequalities $|f_k(z)| \leq a_k$ hold for every k and for every point $z \in D$, then the series $\sum_{k=1}^{\infty} f_k(z)$ converges uniformly in the region D.*

THEOREM 1.5. *A uniformly convergent series consisting of continuous functions converges to a continuous function.*

1) In connection with these theorems see also subsection 2 of §5 and subsection 3 of §8.

2. Holomorphic functional element. Condition of Cauchy-Riemann.

DEFINITION (*Holomorphic functional element*). The function

$$f = f(z_1, \cdots, z_n) = u(z_1, \cdots, z_n) + iv(z_1, \cdots, z_n)$$

(where $u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$), given in a region D of the space C^n, is holomorphic in that region, or, in other words, constitutes a holomorphic functional element, if it has at every point $z \in D$ partial derivatives

$$\frac{\partial f}{\partial z_k} = \lim_{\Delta z_k \to 0} \frac{f(z_1, \cdots, z_{k-1}, z_k + \Delta z_k, z_{k+1}, \cdots, z_n) - f(z_1, \cdots, z_n)}{\Delta z_k}, \quad (1.1)$$

where $k = 1, \cdots, n$. We observe that in this definition (see subsection 1 of the present section) the function f is supposed single-valued. One says that the function f is *holomorphic* at the point $z \in C^n_z$, or, in other words, is a *holomorphic functional element* at the point z, if it is holomorphic in some neighborhood U_z of the point z. The expression "the function f is *regular* at the point z" has exactly the same meaning.

One also says that a function holomorphic at the point z represents at that point a *holomorphic functional germ* (similar terminology is used in the theory of bundles – see subsections 2 and 5 of §4).

A function $f(z)$ holomorphic at every point of some set $E \subset C^n$ is said to be holomorphic on that set.

Take as a neighborhood of some point $z^{(0)}$ $(z_1^{(0)}, \cdots, z_n^{(0)})$ the region $U = \{ | z_k - z_k^{(0)} | < R_k,\ k = 1, \cdots, n\}$. This region will be called a *circular polycylinder* or simply *polycylinder* (for $n = 2$ a circular bicylinder or simply a bicylinder) with center at the point $z^{(0)}$ (in connection with these definitions see also subsection 1 of §2). In view of the given definition the function $f(z_1, \cdots, z_n)$ is said to be holomorphic in the region U if each of the functions $f(z_1^{(0)}, \cdots, z_{k-1}^{(0)}, z_k, z_{k+1}^{(0)}, \cdots, z_n^{(0)})$ is a holomorphic function of its argument in the disk $| z_k - z_k^{(0)} | < R_k$; in other words, the function $f(z_1, \cdots, z_n)$ is said to be holomorphic at the point $z^{(0)}$ or, as is sometimes said, in the set of its arguments, if it is holomorphic at that point in each of its arguments taken separately.

For the function $f(z) = u(z) + iv(z)$ of one complex variable z, where $u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$ are differentiable functions of the real variables x and y $(z = x + iy)$, necessary and sufficient conditions for the existence of the

derivative df/dz are the Cauchy-Riemann conditions. In our case they correspondingly give conditions for the existence of the partial derivatives of the function $f(z_1, \cdots, z_n)$ and have the form

$$\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k}, \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}; \quad k = 1, \cdots, n. \tag{1.2}$$

These conditions are more conveniently written in terms of the so-called formal derivatives. The latter are obtained as follows: suppose that the function $f(x_1, y_1, \cdots, x_n, y_n)$ (possibly assuming complex values) has partial derivatives in all of its variables. We consider its differential

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial y_1} dy_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n + \frac{\partial f}{\partial y_n} dy_n. \tag{1.3}$$

Here, if $f = u + iv$, then $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial u}{\partial x_1} + i \dfrac{\partial v}{\partial x_1}$ and so forth.

We put

$$dz_k = dx_k + idy_k; \quad d\bar{z}_k = dx_k - idy_k$$

(as always $\bar{z}_k = x_k - iy_k$) and thereupon substitute (in (1.3))

$$dx_k = \frac{dz_k + d\bar{z}_k}{2}, \quad dy_k = \frac{dz_k - d\bar{z}_k}{2i}.$$

Then, if we write

$$\frac{\partial f}{\partial z_k} = \frac{1}{2}\left[\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\right]; \quad \frac{\partial f}{\partial \bar{z}_k} = \frac{1}{2}\left[\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right], \tag{1.4}$$

equation (1.3) takes the form

$$df = \frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial f}{\partial z_n} dz_n + \frac{\partial f}{\partial \bar{z}_n} d\bar{z}_n. \tag{1.5}$$

The quantities $\partial f/\partial z_k$, $\partial f/\partial \bar{z}_k$ defined by (1.4) bear the name of *formal derivatives*. This terminology arises because of the possibility of obtaining them formally using the chain differentiation rule and the equations

$$x_k = \frac{z_k + \bar{z}_k}{2}, \quad y_k = \frac{z_k - \bar{z}_k}{2i}.$$

In formal derivatives the Cauchy-Riemann conditions (1.2) may be written

as follows:

$$\frac{\partial f}{\partial \bar{z}_k} = 0, \quad k = 1, \cdots, n. \tag{1.6}$$

If the functions $f(z_1, \cdots, z_n)$ are holomorphic, then their formal derivatives $\partial f / \partial z_k$ coincide with the partial derivatives of the function f defined in (1.1).

Now we formulate the following proposition.

THEOREM 1.6 (*Fundamental theorem of Hartogs*). *If the function $f(z_1, \cdots, z_n)$ is holomorphic in the polycylinder*

$$U\{ | z_k - z_k^{(0)} | \leq R_k, \quad k = 1, \cdots, n\},$$

then it is continuous in that polycylinder.

The difficulties arising in the proof of this proposition are connected with the fact that from the existence of partial derivatives of the function $f(z_1, \cdots, z_n)$ it immediately follows only that the function is continuous in each variable separately. These difficulties were overcome and the theorem proved by F. Hartogs (1874–1943) in 1905. In his proof Hartogs [1] essentially used a result of Osgood, who already in 1900 had arrived at the same conclusion under additional requirements of boundedness of the function $f(z_1, \cdots, z_n)$ in the polycylinder U. Until the paper of Hartogs appeared, the requirement that the function be continuous was added to the requirement of existence of partial derivatives.

The continuity of a holomorphic function in the set of its variables is used to obtain its representation in the form of an n-dimensional Cauchy integral. From the integral representation of Cauchy for polycylindrical regions as considered in the following section we deduce the representation of a holomorphic function $f(z_1, \cdots, z_n)$ in the form of a multiple power series. Thus it will be shown that the definition given by us for a holomorphic functional element is equivalent to its definition as the sum of the corresponding power series.

The following two subsections of the present section are devoted to the proof of Theorem 1.6. We carry out this proof for functions of two complex variables w and z, assuming $w_0 = z_0 = 0$.

3. **Hartogs' lemma.** First of all we prove a lemma which is at the basis of Hartogs' investigations. It is called *the principal theorem*

of Hartogs. [1] It consists of the following:

LEMMA. *Suppose that the function* $f(w, z)$ *is* 1) *holomorphic in the bicylinder* $U_{B,R}\{|w| \leq B, |z| \leq R\}$ *and* 2) *bounded in the bicylinder* $U_{\beta,R}\{|w| \leq \beta, |z| \leq R\}$. *Then the function* $f(w, z)$ *is continuous in the bicylinder* $U_{B,R}$.

PROOF. From the first hypothesis of the lemma and the definition of a holomorphic function of two variables it follows that in the bicylinder $U_{B,R}$ the given function may be represented by the series

$$f(w, z) = \sum_{\nu=0}^{\infty} f_\nu(z)\, w^\nu, \tag{1.7}$$

which is uniformly and absolutely convergent in the disk $|w| \leq B$ for each point z of the disk $|z| \leq R$.

First of all we prove that all the functions $f_\nu(z)$ are holomorphic in the disk $|z| \leq R$. For the function $f_0(z) = f(0, z)$ this follows immediately from the first hypothesis of the lemma. Therefore it is sufficient to establish that for $m \geq 1$ the function $f_m(z)$ is holomorphic in the disk $|z| \leq R$ if the functions $f_0(z), \cdots, f_{m-1}(z)$ are holomorphic there.

Consider the function

$$\psi(w, z) = \frac{f(w, z) - \sum_{\nu=0}^{m-1} f_\nu(z)\, w^\nu}{w^m} = \sum_{k=\nu}^{\infty} f_{m+k}(z)\, w^k. \tag{1.8}$$

Evidently for each w, $0 < |w| \leq B$, the function $\psi(w, z)$ is holomorphic in z in the disk $|z| \leq R$, while

$$\lim_{w \to 0} \psi(w, z) = f_m(z).$$

We shall show that this limit is attained uniformly in z in the disk $|z| \leq R$. In view of the second hypothesis of the lemma there exists a number $g > 0$ such that in the bicylinder $U_{\beta,R}$,

$$|f(w, z)| < g.$$

1) One may find a generalization of the principal theorem of Hartogs in the paper of Rothstein [1]. A number of consequences of Rothstein's results were obtained by Sakai [1].

The Cauchy inequality for the series (1.7) yields the fact that in the disk $|z| \leq R$,

$$|f_\nu(z)| < \frac{g}{\beta^\nu} \quad (\nu = 0, 1, 2, \cdots). \tag{1.9}$$

Now from (1.8) and (1.9) we have (on the assumption that $|w| < \beta_1$, where $0 < \beta_1 < \beta$)

$$|\psi(w, z) - f_m(z)| = \left|\sum_{k=1}^{\infty} f_{m+k}(z)\, w^k\right| \leq$$

$$\leq \sum_{k=1}^{\infty} |f_{m+k}(z)\, w^k| \leq \sum_{k=1}^{\infty} g\beta^{-m}\left(\frac{|w|}{\beta}\right)^k = \frac{g\,|w|}{\beta^m(\beta - |w|)}.$$

This last expression does not depend on z. Therefore one immediately perceives that the limit process in question is uniform and accordingly that the function $f_m(z)$ is holomorphic in the disk $|z| \leq R$.

Choose a number $r < R$, and draw a circle $|z| = R_1$ (where $r < R_1 < R$) so that none of the zeros of the functions $f_\nu(z)$, $\nu = 0, 1, \cdots$, lie on it. The set of those zeros is no more than countable, and therefore such a circle may always be constructed. In view of (1.9) in the disc $|z| \leq R$,

$$\left|\frac{f_\nu(z)}{g} B^\nu\right| < \left(\frac{B}{\beta}\right)^\nu. \tag{1.10}$$

Denote by Q_ν the set of those points of the circle $|z| = R_1$ for which the left side of inequality (1.10) is larger than unity. We consider the sequence of sets Q_ν, $\nu = 0, 1, \cdots$. Because of the convergence of the series (1.7) for $|z| = R_1$ and $|w| = B$, there are on the circle $|z| = R_1$ no points belonging to the infinite collection of sets Q_ν. Therefore, if a_ν is the measure of the set Q_ν, then $\lim_{\nu\to\infty} a_\nu = 0$.[1]

Consider in the disc $|z| \leq R_1$ the function

$$h_\nu(z) = \frac{1}{\nu}\ln\left|\frac{f_\nu(z)}{g}\right| + \ln B \quad (\nu = 1, 2, \cdots). \tag{1.11}$$

It is harmonic everywhere in the disk $|z| \leq R_1$ with the possible exception of a

1) The measurability of the set Q_ν is obvious. For further details see e.g. the original article of F. Hartogs in Math. Ann. 62 (1906).

finite set of points which are zeros of the function $f_\nu(z)$, on the approach to which $h_\nu(z) \to -\infty$. We shall regard $-\infty$ as the value of the function $h_\nu(z)$ at these points. Then, using the Poisson integral, we construct a harmonic function $p_\nu(z)$ equal to $\ln(B/\beta)$ on the set Q_ν and equal to zero on the remainder of the circle $|z| = R_1$.[1] In view of (1.10) and (1.11)

$$h_\nu(z) < p_\nu(z) \tag{1.12}$$

in the entire disk $|z| < R_1$. Applying Poisson's formula, we further obtain for the points z of that disk

$$p_\nu(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_\nu(R_1 e^{i\theta}) \frac{R_1^2 - |z|^2}{R_1^2 + |z|^2 - 2R|z|\cos(\theta - \arg z)}\, d\theta \leq$$

$$\leq \frac{1}{2\pi} \frac{R_1 + |z|}{R_1 - |z|} (\ln B - \ln \beta)\, a_\nu. \tag{1.13}$$

In order to obtain inequality (1.13) we have to break the Poisson integral into two parts (over the set Q_ν and over the remaining portion of the circle $|z| = R_1$) and then use the inequality

$$\int_E v\, dx \leq M \operatorname{mes} E,$$

where M is the upper bound of the function v on the set E, and $\operatorname{mes} E$ denotes the measure of the set E. Since $\lim_{\nu\to\infty} a_\nu = 0$, it follows from (1.13) that in the disk $|z| \leq r$ for $\nu > N(\epsilon)$

$$p_\nu(z) < \epsilon$$

(here ϵ is any positive number and the quantity $N(\epsilon)$ is chosen in terms of ϵ), and accordingly in view of (1.12)

$$h_\nu(z) = \frac{1}{\nu} \ln \left| \frac{f_\nu(z)}{g} \right| + \ln B < \epsilon$$

or

---

1) As Carathéodory remarks (Carathéodory, *Funktionentheorie*, Vol. II, p. 107, Verlag Birkhäuser, Basel, 1950) the construction and use of the function $p_\nu(z)$ by Hartogs in the course of the present proof was the first case of the application of the concept of harmonic measure. In the process of the further development of the theory of functions the concept of harmonic measure was rediscovered and widely used.

$$|f_\nu(z)| < g\left[\frac{e^\epsilon}{B}\right]^\nu.$$

Hence it follows that the series (1.7) converges uniformly in the bicylinder $\{|w| < B/e^\epsilon, |z| \le r\}$ and accordingly the function $f(w, z)$ is continuous in this bicylinder. The number ϵ may be chosen as small as desired, and the number r as close as desired to R. Hence it follows that the function $f(w, z)$ is continuous at every interior point of the closed bicylinder $U_{B,R}$.

From the definition of a holomorphic function, in the first hypothesis of the lemma the closed bicylinder $U_{B,R}$ may be replaced by another bicylinder with center at the origin of coordinates somewhat larger in size than $U_{B,R}$. Hence it follows that the continuity of the function $f(w, z)$ holds in all the points of the closed bicylinder $U_{B,R}$.

4. Osgood's lemma. Completion of the proof of the fundamental lemma of Hartogs. Now we shall prove the following proposition due to Osgood.

LEMMA. *If the function $f(w, z)$ is holomorphic in the bicylinder $U\{|w| \le S, |z| \le R\}$, then in some region $D \subset U$ the function $f(w, z)$ is bounded.*

PROOF. Suppose that w_0 is some point of the disk $|w| < S$. We consider the function $f(w_0, z)$ for $|z| \le R$. We denote by $M(w_0)$ the upper bound of the values of the modulus of the function in this disk. Further we denote by P_n the set of points w in the disk $|w| < S$ for which

$$M(w) \le n.$$

Evidently, 1) $P_n \subset P_{n+1}$; 2) every point w of the disk $|w| < S$ belongs to the set P_n, beginning with some number n. It is also easy to see that 3) the set P_n is closed. Indeed, for each fixed point z_0, where $|z_0| \le R$, the function $f(w, z_0)$ is continuous in the disk $|w| < S$. Therefore, if $|f(w_i, z_n)| \le n$, where $i = 1, 2, \cdots$, $\lim_{i\to\infty} w_i = w^*$, $|w^*| < S$, then also $|f(w^*, z_0)| \le n$. Therefore our assertion follows with respect to the set P_n.

Now we shall show that there exists a set P_N containing some region τ which is a portion of the disk $|w| < S$. Indeed, in the contrary case the set P_n would be nowhere dense in the disk $|w| < S$, and in every region σ which is a portion of that disk we could find a disk σ_1 inside of which and on the boundary of which there would be no points of the set P_1 (if this were not possible, then because of the fact that P_1 is closed the entire region σ would belong to P_1).

In σ_1 we select in the same way a disk σ_2 entirely (along with its boundary) free of points of the set P_2. Carrying out this process, we obtain a sequence of disks $\sigma_1, \sigma_2, \cdots$; each of these disks lies inside the preceding one; therefore they have at least one common point, which does not belong to any of the sets P_n. Since this is impossible, there exists a region τ, a portion of the disk $|w| < S$, at whose points, for each $|z| \leq R$, we will have

$$|f(w, z)| \leq N.$$

Now suppose that the point $w_0 \in \tau$. Then since τ is a region, there exists a disk $\{|w - w_0| \leq \rho\} \subset \tau$. We have proved that the function $f(w, z)$ is bounded in the bicylinder $D\{|w - w_0| \leq \rho, |z| \leq R\} \subset U$.

COMPLETION OF THE PROOF OF THEOREM 1.6. Suppose that the bicylinder U in the formulation of Theorem 1.6 for the case of two variables w and z is replaced by the bicylinder $U\{|w| \leq S, |z| \leq R\}$, in which $f(w, z)$ is given as a holomorphic function. Then this function will in particular be holomorphic in the bicylinder $U_1\{|w| \leq S/3, |z| \leq R\}$. Applying Osgood's lemma, we obtain a polycylinder $V_1\{|w - w_0| \leq \rho, |z| \leq R\}$, in which the function $f(w, z)$ turns out to be bounded. Since $|w_0| < S/3$, the function $f(w, z)$ is holomorphic in the bicylinder $V = \{|w - w_0| < 2S/3, |z| \leq R\}$, constituting a portion of the bicylinder U. Applying Hartogs' lemma to the bicylinders V_1 and V with centers at the point $(w_0, 0)$, we find that the function $f(w, z)$ is continuous in the bicylinder V. This bicylinder is closed. Accordingly the function $f(w, z)$ is bounded in it, and also in the bicylinder $U_1 \subset V$. Applying Hartogs' lemma to the bicylinders U_1 and U, we complete the proof of Theorem 1.6.

5. **Integration of functions of a complex variable.** We begin by considering a k-dimensional l-smooth (where $l \geq 1$) manifold V reducing to a single point. Thus, we suppose that in the structural atlas of the manifold V there is a chart (V, ψ), where $\psi V = W$ is a solid sphere (or simplex) in the space of local coordinates $t_1, \cdots, t_k$. The region of application of this system of coordinates in the case singled out is the entire manifold V.

Suppose that $f(t_1, \cdots, t_k)$ is an integrable function (in the sense of Lebesgue) defined on a closed region $\overline{W}$. We set up an odd differential form of degree k,

$$\alpha = a_{12\ldots k}\, dt_1 \wedge \cdots \wedge dt_k, \quad a_{12\ldots k} = f(t_1, \cdots, t_k). \tag{1.14}$$

The integral of the odd form α over the manifold V is defined by the equation

$$\int_V \alpha = \int_W a_{12\ldots k}\, dt_1 \cdots dt_k. \tag{1.15$_1$}$$

Evidently the value $\int_V \alpha$ does not depend on the choice of the local system of coordinates. This results from the transformation rule for the components of an odd covector (0.3_2) from one coordinate system to another and from the formula for substituting variables under a multi-dimensional integral sign.

Now we consider an oriented element of the manifold V. Let ϵ be its orientation. We put the orientation of the homeomorphism ψ equal to unity. Then the number ϵ will be at the same time the orientation of the region $W = \psi V$ in the space of local coordinates $t_1, \cdots, t_k$.

We will further consider integrals of two types of the differential form α over an oriented element of the manifold V. In the integral of the first type the form α defined by equation (1.14) is taken to be odd and the integral $\int_V \alpha$ is defined by equation (1.15_1). In the integral of the second type the form α, given by equation (1.14), is taken to be even (the difference between even and odd forms appears on passing to other systems of local coordinates); the integral $\int_V \alpha$ in this case is defined by the equation

$$\int_{V^\epsilon} \alpha = \int_{W^\epsilon} \alpha = \int_W \epsilon\, a_{12\ldots k}\, dt_1 \cdots dt_k. \tag{1.15$_2$}$$

The integral (1.15_1) may be computed also for even forms, and the integral (1.15_2) for odd forms. However these integrals turn out to be equal to zero (since only in this case do their values turn out not to depend on the choice of the local system of coordinates).

In the integrals below, indications as to the evenness or oddness of forms α, or of the presence or absence of orientation for the element V (and accordingly the index ϵ in the notations V^ϵ, W^ϵ), will be omitted whenever no ambiguity can arise. One need only remember that an integral over a nonorientable manifold always involves an odd form α.

Now we shall suppose that V is a surface element (curvilinear simplex) given by equations (0.1_5) in the space R_N of real variables $x_1, \cdots, x_N$, and that α is an even or odd form of degree k defined on the set $U \subset R_N$, where $V \subset U$, by the equation

$$\alpha = \sum_k^N{}' a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{1.16}$$

where the $a_{i_1 \cdots i_p}$ are functions of the coordinates $x_1, \cdots, x_N$. In the space R_N we consider an orientation introduced with the aid of this system of coordinates.

Equations (0.1) define an inclusion μ of an element of the manifold V into the space R_N. Using formula (0.8) or (0.9), we represent the form α on the element V in the form

$$\alpha = \left[\sum_k^N{}' a^*_{i_1 \cdots i_k} \frac{\partial (x_{i_1}, \cdots, x_{i_k})}{\partial (t_1, \cdots, t_k)} \right] dt_1 \wedge \cdots \wedge dt_k. \tag{1.16'}$$

This expression is called the *trace* or the restriction of the form α on the element V. We now write α instead of $\alpha \circ \mu$ (since $V \subset R_N$). In order to obtain the expression (1.16′) in the case of an odd form α we need to put the orientation ϵ of the element of the manifold V and the orientation ϵ' of the space R_N equal to unity; $a^*_{i_1 \cdots i_k}$ is the result of replacing the variables $x_1, \cdots, x_N$ in the quantities $a_{i_1 \cdots i_k}$ by their expressions in terms of $t_1, \cdots, t_k$ from formula (0.1_5).

Now we put [1]

$$\int_V \alpha = \int_V \sum_k^N{}' a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} =$$

$$= \int_V \left[\sum_k^N{}' a^*_{i_1 \cdots i_k} \frac{\partial (x_{i_1}, \cdots, x_{i_k})}{\partial (t_1, \cdots, t_k)} \right] dt_1 \wedge \cdots \wedge dt_k =$$

$$= \int_W \epsilon \left[\sum_k^N{}' a_{i_1 \cdots i_k} \frac{\partial (x_{i_1}, \cdots, x_{i_k})}{\partial (t_1, \cdots, t_k)} \right] dt_1 \cdots dt_k. \tag{1.17}$$

In the case of an integral over a nonoriented element of V we must put $\epsilon = 1$. If the element of V is oriented, then ϵ is its orientation. In both cases the latter integral (1.17) is taken over the nonoriented region W.

This definition of integral easily extends to the case of a k-dimensional piecewise smooth surface $\mathcal{V}$, where $\mathcal{V} = \bigcup_{s=1}^m V^{(s)}$ and the $V^{(s)}$ are surface elements, namely curvilinear simplexes of the sort described above. We put

1) The symbols for exterior multiplication of differentials under the integral sign are sometimes not written but are always implicit.

$$\int_{\mathcal{V}} \alpha = \sum_{s=1}^{m} \int_{V(s)} \alpha. \tag{1.18}$$

An integral over a manifold $\mathcal{V}$ of a more general sort will be considered below in Chapter IV.

We return to the consideration of the space $C^n = R_N$ (where $N = 2n$) of complex variables $z_p = x_p + ix_{n+p}$, $p = 1, \cdots, n$. Further we will as usual put $x_{n+p} = y_p$. We replace equations (0.1) defining the surface element V by equations of the type

$$z_p = z_p(t_1, \cdots, t_k), \quad p = 1, \cdots, n. \tag{1.19}$$

Using the relations

$$dx_p = \frac{1}{2}(dz_p + d\bar{z}_p), \quad dy_p = -\frac{i}{2}(dz_p - d\bar{z}_p),$$

we obtain for the form α, which may now be either even or odd, the following expression (instead of (1.16)):

$$\alpha = \sum_{k}^{2n}{}' A_{i_1 \cdots i_k} d\zeta_{i_1} \wedge \cdots \wedge d\zeta_{i_k}. \tag{1.20}$$

Here $\zeta_p = z_p$ for $1 \le p \le n$, $\zeta_p = \overline{z_{p-n}}$ for $n < p \le 2n$. The corresponding rule may be established for the replacement of the indices in the quantities $A_{i_1 \cdots i_k}$. Thus, for example, for $n = 6$ we have: $A_{124} = A_{12\bar{1}}$, $A_{235} = A_{23\bar{2}}$ and so forth.

In order to obtain the quantities $A_{j_1 \cdots j_k}$ from the quantities $a_{i_1 \cdots i_k}$ one must apply the formula

$$A_{j_1 \cdots j_k} = \eta \sum_{k}^{2n}{}' a_{i_1 \cdots i_k} \frac{\partial(x_{i_1}, \cdots, x_{i_k})}{\partial(\zeta_{j_1}, \cdots, \zeta_{j_k})}. \tag{1.21}$$

Here $\eta = 1$ for an even covector and $\eta = i^n$ for an odd covector; the formal derivative $\partial x_p / \partial \zeta_q$ must be calculated according to formulas (1.4). The components of the covector $A_{j_1 \cdots j_k}$ under passage in the space C^n from one complex coordinate system $z_1, \cdots, z_n$ to another transform according to formulas (0.3_1) or (0.3_2). However, the derivatives entering into these formulas must be calculated according to the rules of formal differentiation (1.4). The form

$$\alpha = \sum_{p}^{n}{}' A_{i_1 \cdots i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \tag{1.20$_1$}$$

where $A_{i_1 \cdots i_p}(z)$ are holomorphic functions in some region $D \subset C^n$, is said to be *holomorphic* in that region D. After these alterations we obtain in place of formula (1.17):

$$\int_V \sum_{k}^{2n}{}' A_{i_1 \cdots i_k} d\zeta_{i_1} \wedge \cdots \wedge d\zeta_{i_k} =$$

$$= \int_V \left[\sum_{k}^{2n}{}' A^*_{i_1 \cdots i_k} \frac{\partial(\zeta_{i_1}, \cdots, \zeta_{i_k})}{\partial(t_1, \cdots, t_k)} \right] dt_1 \wedge \cdots \wedge dt_k =$$

$$= \int_W \epsilon \left[\sum_{k}^{2n}{}' A^*_{i_1 \cdots i_k} \frac{\partial(\zeta_{i_1}, \cdots, \zeta_{i_k})}{\partial(t_1, \cdots, t_k)} \right] dt_1 \cdots dt_k. \tag{1.22}$$

Formula (1.18) remains unchanged.

In what follows we shall most frequently employ the following special cases of the integral (1.22):

1) $k = 1$; the integral (1.22) is taken over a piecewise smooth curve.

2) All the components of the covector $A_{i_1 \cdots i_k}$ for which at least one index $i_p > n$ are equal to zero. In this case, in place of (1.20) we obtain

$$\alpha = \sum_{k}^{n}{}' A_{i_1 \cdots i_k} dz_{i_1} \wedge \cdots \wedge dz_{i_k}; \tag{1.20$_2$}$$

in place of (1.22) we get

$$\int_V \sum_{k}^{n}{}' A_{i_1 \cdots i_k} dz_{i_1} \wedge \cdots \wedge dz_{i_k} =$$

$$= \int_V \left[\sum_{k}^{n}{}' A^*_{i_1 \cdots i_k} \frac{\partial(z_{i_1}, \cdots, z_{i_k})}{\partial(t_1, \cdots, t_k)} \right] dt_1 \wedge \cdots \wedge dt_k =$$

$$= \int_W \epsilon \left[\sum_{k}^{n}{}' A^*_{i_1 \cdots i_k} \frac{\partial(z_{i_1}, \cdots, z_{i_k})}{\partial(t_1, \cdots, t_k)} \right] dt_1 \cdots dt_k. \tag{1.22$_1$}$$

In the special case when $k = n$ we may by putting $A_{1 \cdots n} = f(z)$ bring the formula (1.22$_1$) into the following form:

$$\int_V f dz_1 \wedge \cdots \wedge dz_n = \int_V f dz_1 \cdots dz_n =$$
$$= \int_W \epsilon f \frac{\partial(z_1, \cdots, z_n)}{\partial(t_1, \cdots, t_n)} dt_1 \cdots dt_n. \tag{1.23}$$

Now suppose that equations (1.19) have the form

$$z_p = z_p(t_p), \; p = 1, \cdots, n,$$

and that each of these functions is continuous, has a continuous derivative on the closed interval $a_p \leq t_p \leq b_p$, and realizes a homeomorphism of this interval onto the curve $\Gamma_p \subset C^1_{z_p}$. Then $V = \Gamma_1 \times \cdots \times \Gamma_n$ is called the Cartesian product of the curves $\Gamma_1, \cdots, \Gamma_n$. For this case, beginning with formula (1.23), one may obtain after some calculation (here omitted) the formula

$$\int_V f dz_1 \cdots dz_n = \int_{\alpha_1 \Gamma_1}^{\beta_1} dz_1 \cdots \int_{\alpha_n \Gamma_n}^{\beta_n} f dz_n, \tag{1.24}$$

where $\alpha_p = z_p(a_p)$, $\beta_p = z_p(b_p)$.

§2. CAUCHY INTEGRAL FORMULA FOR POLYCYLINDRICAL REGIONS. FUNDAMENTAL PROPERTIES OF A HOLOMORPHIC FUNCTIONAL ELEMENT

1. Polycylindrical regions. Let D_k be some region of the plane of the variable z_k, $k = 1, \cdots, n$. The collection of all points $z \in C^n$ with coordinates $z_1, \cdots, z_n$ satisfying the condition $z_k \in D_k$, $k = 1, \cdots, n$, forms a polycylindrical region of the space C^n of the variables $z_1, \cdots, z_n$. It is the Cartesian product of the regions D_k and is denoted by the symbol $D = D_1 \times \cdots \times D_n$. If the regions $D_k = \{ | z_k - a_k | < r_k \}$, then as in the preceding section the region D is called a *circular polycylinder with center at the point* $(a_1, \cdots, a_k)$ or simply a *polycylinder*. In the case when all $r_k = r$ the corresponding polycylinder is called a *polycylinder of radius* r; if $r = 1$ it is said to be a *unit polycylinder*. In the space of two variables w, z, we will correspondingly consider bicylindrical regions, bicylinders, and the unit bicylinder.

From general topological arguments it follows that if D_k is a simply-connected region, then the region D is homeomorphic to the solid sphere of the corresponding number of variables.

Further it is clear that the boundary of the polycylindrical region D consists of points $(z_1, \cdots, z_n)$ one coordinate of which $z_k \in \partial D_k$, while the remainder $z_s \in \overline{D}_s$ $(s \neq k)$. The portion of the boundary ∂D consisting of all such points for a fixed k will be denoted by $D^{(k)}$. Thus, $\partial D = \Sigma_{k=1}^n D^{(k)}$. The portion of ∂D whose points simultaneously belong to all of the $D^{(k)}$ is particularly important. This portion is denoted by S and is called the *skeleton of the boundary* of the polycylindrical region.

In the case when the boundaries ∂D_k are piecewise smooth curves, the boundaries $D^{(k)}$ are $(2n-1)$-dimensional surfaces, and the skeleton S of the boundary is the set of n-dimensional surfaces constituting their intersections. In this case the polycylindrical region D is said to be *ordinary*. The surfaces making up S form, so to speak, the n-dimensional edges of the boundary.

REMARK. The skeleton of the boundary S of the unit bicylinder $E\{|w| < 1, |z| < 1\}$ (and accordingly also every other bicylindrical region which is the product of simply-connected regions) is homeomorphic to the torus. On the surface $S\{|w| = 1, |z| = 1\}$ one may construct two sections (for example, by the circles $\{w = 1, |z| = 1\}$ and $\{|w| = 1, z = 1\}$), after which it none the less remains connected.

2. Cauchy integral formula.

THEOREM 2.1. *If the function $f(z)$ is holomorphic in a bounded ordinary polycylindrical region D, and continuous in the closed region $\overline{D}$, then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_S \frac{f(t)}{(t_1 - z_1) \cdots (t_n - z_n)} \, dt_1 \wedge \cdots \wedge dt_n. \tag{1.25}$$

REMARK. The continuity of the function $f(z)$ inside the region D follows from its holomorphicity in view of the fundamental theorem of Hartogs.

PROOF. For brevity we shall restrict ourselves to the case of two variables w, z. The points of the sets $D^{(1)}$ and $D^{(2)}$ may be regarded as limit points of interior points of the region D. Suppose further that (w, z) denotes a point of the region D, (t_1, z) a point of $D^{(1)}$, and (w, t_2) a point of $D^{(2)}$. Then one may assume that

$$\lim_{w \to t_1} f(w, z) = f(t_1, z), \quad \lim_{z \to t_2} f(w, z) = f(w, t_2).$$

The function $f(w, z)$ is uniformly continuous in the closed region $\overline{D}$, and these

limits are taken on uniformly (the first with respect to z, the second with respect to w). It therefore follows, as a consequence of a theorem of Weierstrass[1] for the case of one variable, that $f(t_1, z)$ is a holomorphic function of z in the region D_2 and that $f(w, t_2)$ is a holomorphic function of w in the region D_1. Applying Cauchy's integral formula for one variable, we obtain

$$\left.\begin{aligned} f(w, z) &= \frac{1}{2\pi i} \int_{\partial D_1} \frac{f(t_1, z)}{t_1 - w}\, dt_1, \\ f(t_1, z) &= \frac{1}{2\pi i} \int_{\partial D_2} \frac{f(t_1, t_2)}{t_2 - z}\, dt_2. \end{aligned}\right\} \tag{1.26}$$

Here the paths of integration ∂D_1 and ∂D_2 are traversed in the usual direction. From these equations, on replacing the repeated integral by a double integral, we obtain from formula (1.24)

$$f(w, z) = \frac{1}{(2\pi i)^2} \int_{\partial D_1} \frac{dt}{t_1 - w} \int_{\partial D_2} \frac{f(t_1, t_2)}{t_2 - z}\, dt_2 =$$

$$= \frac{1}{(2\pi i)^2} \int_S \frac{f(t_1, t_2)}{(t_1 - w)(t_2 - z)}\, dt_1 \wedge dt_2. \tag{1.25$_1$}$$

The surface S is assumed to be so oriented that on replacing the double integral by the corresponding repeated integral the integrals (1.26) turn out to be oriented in the way described above. In an analogous way the surface S is assumed to be oriented in the general case of n variables.

REMARK. In the derivation of the Cauchy integral formula for a polycylindrical region we may suppose that the regions D_k are bounded by arbitrary rectifiable curves. Without changing the results in any essential way, this leads to additional computational difficulties in a number of cases.

From the Cauchy integral formula (1.25) it follows that *a function* $f(z)$, *holomorphic in an ordinary bicylindrical region* D, *is determined by its values on the skeleton* S *of its boundary.*

In subsection 4 of the following section it is proved that if the function $f(z)$ is holomorphic in the region D and continuous in the closed region D,

1) An analogous corollary of the Weierstrass theorem for the case of a function of two variables is derived in the following section.

then $|f(z)|$ takes on its maximum value on the boundary ∂D of the region D. If D is an ordinary polycylindrical region, then (assuming that $f(z)$ is not constant) the magnitude $|f|$ will take on its largest value on the skeleton S of the boundary. For the case of two variables this follows from the fact that the function $f(w, t_2)$ is holomorphic in w in the region D for all $t_2 \in \partial D_2$ and the function $f(t_1, z)$ is holomorphic in z in the region D_2 for all $t_1 \in \partial D_1$.

In what follows we will encounter other classes of regions for which one may single out a part of the boundary with analogous properties. Such a subset of the boundary is called *the boundary of a region in the sense of Šilov* relative to the class of functions holomorphic in that region and continuous in the closed region.

In conclusion we note that just as in the theory of functions of one variable we may here consider integrals "of Cauchy type." If on the skeleton S of the boundary of some ordinary polycylindrical region D there is given a continuous function $\phi(t)$, then the integral

$$\frac{1}{(2\pi i)^n} \int_S \frac{\phi(t)}{(t_1 - z_1)\cdots(t_n - z_n)} dt_1 \wedge \cdots \wedge dt_n = f(z) \tag{1.27}$$

defines in the region D a holomorphic functional element $f(z)$. However, generally speaking, its values do not converge to the values of the function $\phi(t)$ as the point $z \in D$ tends toward the skeleton S. The integral (1.27) is called an integral of Cauchy type. The holomorphicity of the function $f(z)$ in the region D is proved in the same way as the analogous proposition in the theory of functions of one variable.

Integrals of Cauchy type were investigated in detail by V. A. Kakičev [1]. We shall mention one of his results, restricting ourselves for simplicity to the case of two variables w, z. Suppose that $D = D_1 \times D_2$ is an ordinary bicylindrical region and that $\phi(t_1, t_2)$ is a function defined on the skeleton S of this region and satisfying the Hölder condition

$$|\phi(t_1, t_2) - \phi(\tau_1, \tau_2)| \leq A_1 |t_1 - \tau_1|^{\alpha_1} + A_2 |t_2 - \tau_2|^{\alpha_2},$$

where A_k is some constant, $0 \leq \alpha_k \leq 1$, $k = 1, 2$. Consider an integral of Cauchy type (1.27) which defines the functions $f^{++}(w, z)$, $f^{--}(w, z)$, $f^{+-}(w, z)$, $f^{-+}(w, z)$, holomorphic respectively in the regions $D_1 \times D_2$, $D_1^- \times D_2^-$, $D_1 \times D_2^-$, $D_1^- \times D_2$, and continuous in these closed regions. Here D_k^- is the region complementary to the region D_k in the plane of the complex variable z_k. We have

THEOREM 2.2. *On the skeleton S of an ordinary bicylindrical region $D_1 \times D_2$ the following equations hold:*

$$f^{++} + f^{+-} + f^{-+} + f^{--} = -\frac{1}{\pi^2} \int_S \frac{\phi(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)} d\tau_1 \wedge d\tau_2;$$

$$f^{++} - f^{+-} + f^{-+} - f^{--} = \frac{1}{\pi i} \int_{\partial D_1} \frac{\phi(\tau_1, t_2)}{\tau_1 - t_1} d\tau_1;$$

$$f^{++} + f^{+-} - f^{-+} - f^{--} = \frac{1}{\pi i} \int_{\partial D_2} \frac{\phi(t_1, \tau_2)}{\tau_2 - t_2} d\tau_2;$$

$$f^{++} - f^{+-} - f^{-+} + f^{--} = \phi(t_1, t_2).$$

Here all the values of the functions on the left sides of the equations are calculated at an arbitrary point $(t_1, t_2) \in S$.

Theorem 2.2 is a generalization of the well-known theorem of Sohockiĭ from the theory of functions of one variable. We omit the proof.

3. Existence and continuity of all partial derivatives of a holomorphic functional element. The desired existence and continuity follow from the Cauchy integral formula. In the same way as for functions of one variable, one may show that

$$\frac{\partial^{k_1 + \cdots + k_n} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} = \frac{k_1! \cdots k_n!}{(2\pi i)^n} \int_S \frac{f(t)\, dt_1 \wedge \cdots \wedge dt_n}{(t_1 - z_1)^{k_1} \cdots (t_n - z_n)^{k_n}}. \tag{1.28}$$

Therefore our assertion follows.

We note that from equation (1.28) follows the theorem on the possibility of changing the order of differentiation of a holomorphic function of several variables.

4. Differential of a function. For brevity we restrict ourselves to the case of two variables. Using the continuity of the partial derivatives f'_w, f'_z for a holomorphic functional element, we easily obtain

$$\Delta f(w, z) = f'_w(w, z)\Delta w + f'_z(w, z)\Delta z + \eta_1 \Delta w + \eta_2 \Delta z, \tag{1.29}$$

where $\lim \eta_1 = \lim \eta_2 = 0$ for $\Delta w, \Delta z \to 0$. We define the differential of the function $f(w, z)$ by the equation

$$df = f'_w\, dw + f'_z\, dz, \tag{1.30}$$

where $dw = \Delta w$, $dz = \Delta z$. From formulas (1.29) and (1.30) it follows that

$$\Delta f = df + \eta_1 \Delta w + \eta_2 \Delta z.$$

We note that from formula (1.29), just as in the real case, there follows the theorem on the holomorphicity of a composite function and the chain rule for differentiation. If $F = F(w, z)$, $w = w(\zeta)$, $z = z(\zeta)$ are holomorphic functions of their variables, then $F(w(\zeta), z(\zeta))$ is also a holomorphic function of ζ and

$$\frac{dF}{d\zeta} = \frac{\partial F}{\partial w}\frac{dw}{d\zeta} + \frac{\partial F}{\partial z}\frac{dz}{d\zeta}.$$

An analogous rule for differentiation is obtained if $F = F(w, z)$ is a holomorphic function of the complex variables w and z, and $w = w(t)$, $z = z(t)$ are differentiable functions of the real variable t.

5. **Implicit functions.** In complex analysis we can prove various theorems on implicit functions which are analogous to the corresponding theorems of real analysis.

THEOREM 2.3_1. *If the function* $F(w, z) = F(w, z_1, \cdots, z_n)$ *is holomorphic at the point* $(b, a) = (b, a_1, \cdots, a_n)$, $F(b, a) = 0$, $F'_w(b, a) \neq 0$, *then there exists one and only one function* $w = \phi(z) = \phi(z_1, \cdots, z_n)$ *holomorphic at the point* $z = a$ *and satisfying in a neighborhood of that point the relation* $F(\phi(z), z) = 0$.

THEOREM 2.3_2. *If the functions* $F_k(w_1, \cdots, w_p, z_1, \cdots, z_n) = F_k(w, z)$, $k = 1, \cdots, p$, *are holomorphic at the point* $(b, a) = (b_1, \cdots, b_p, a_1, \cdots, a_n)$, $F_k(b, a) = 0$, $[\partial(F_1, \cdots, F_p)/\partial(w_1, \cdots, w_p)]_{(b, a)} \neq 0$, *then there exists one and only one system of functions* $w_k = \phi_k(z) = \phi_k(z_1, \cdots, z_n)$ $(k = 1, \cdots, p)$ *holomorphic at the point* $z = a$ *and satisfying in some neighborhood of that point the relations*

$$F_k(\phi_1(z), \cdots, \phi_p(z), z_1, \cdots, z_k) = 0 \quad (k = 1, \cdots, p).$$

The proof of these theorems is carried out by passing from the complex equations $F_k(w, z) = U_k(u, v, x, y) + iV_k(u, v, x, y) = 0$, where $w = u + iv$, $z = x + iy$, to the real equations $U_k(u, v, x, y) = 0$, $V_k(u, v, x, y) = 0$ $(k = 1, \cdots, p)$. We apply to them the theorems on the existence of implicit functions from real analysis, making use in this connection of the equation

$$\left|\frac{\partial(F_1, \cdots, F_p)}{\partial(w_1, \cdots, w_p)}\right|^2 = \frac{\partial(U_1, V_1, \cdots, U_p, V_p)}{\partial(u_1, v_1, \cdots, u_p, v_p)}. \tag{1.31}$$

Then we verify that the Cauchy-Riemann conditions are satisfied for the functions $w_k = \phi_k(z) = u_k(x, y) + iv_k(x, y)$.

In complex analysis the usual rule for the differentiation of implicit functions is preserved.

6. Pluriharmonic functions. If $f(z) = U(x, y) + iV(x, y)$ is a holomorphic function in the region $D \subset C^n_z$, $U = \operatorname{Re} f$, $V = \operatorname{Im} f$, then, beginning with the Cauchy-Riemann conditions (1.2) and the fact of the existence of successive derivatives of the function f (and therefore of U and V), we easily find ($z_k = x_k + iy_k$; $k, l = 1, \cdots, n$) that

$$\frac{\partial^2 U}{\partial x_k \partial x_l} + \frac{\partial^2 U}{\partial y_k \partial y_l} = 0; \quad \frac{\partial^2 U}{\partial x_k \partial y_l} - \frac{\partial^2 U}{\partial x_l \partial y_k} = 0, \tag{1.32}$$

or in formal derivatives,

$$\frac{\partial^2 U}{\partial z_k \partial \bar{z}_l} = 0. \tag{1.32'}$$

The same equations are satisfied by the function $V(x_1, y_1, \cdots, x_n, y_n)$. The function satisfying these equations will be called a pluriharmonic or polyharmonic function. Evidently every pluriharmonic function is a harmonic function of its variables. Two pluriharmonic functions U and V are said to be *associated* if $U + iV$ is a holomorphic functional element. In this case

$$V(x_1, y_1, \cdots, x_n, y_n) = \int_{P_0}^{P} \sum_{k=1}^{n} \left(-\frac{\partial U}{\partial y_k} dx_k + \frac{\partial U}{\partial x_k} dy_k\right),$$

where the line integral is taken over a path L extending from some fixed point P_0 to the point $P(z_1, \cdots, z_n)$. This last integral does not depend on the path of integration L provided that D is a simply-connected region, that the second derivatives with respect to U are continuous and that the conditions (1.32) are satisfied.[1)] Note that in the case of two variables the conditions (1.32) take

1) The integral $\int_L R_1 dt_1 + \cdots + R_s dt_s$ in the simply-connected region B does not depend on the path of integration if R_k, $\partial R_k/\partial t_l$ is continuous in B and $\partial R_k/\partial t_l = \partial R_l/\partial t_k$.

the form

$$\begin{gathered}\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0, \quad \frac{\partial^2 U}{\partial u^2}+\frac{\partial^2 U}{\partial v^2}=0, \\ \frac{\partial^2 U}{\partial x\,\partial u}+\frac{\partial^2 U}{\partial y\,\partial v}=0, \quad \frac{\partial^2 U}{\partial x\,\partial v}-\frac{\partial^2 U}{\partial y\,\partial u}=0;\end{gathered} \tag{1.33}$$

U in this case is said to be a biharmonic function. [1)]

7. REMARK. From the theorem on the differentiability of a composite function and the definition of the line integral it further follows that if $f(z)$ is a holomorphic functional element in the region D and L is a piecewise smooth curve lying in D, then

$$f(z)-f(z^0)=\int_{(z^0)}^{(z)} \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}\,dz_k, \tag{1.34}$$

where the integral is taken along the curve L.

§3. REPRESENTATION OF A HOLOMORPHIC FUNCTIONAL ELEMENT BY POWER SERIES

1. General propositions. We shall consider (ordinary and multiple) series consisting of holomorphic functions. At the basis of their study there lies, as in the case of one variable, the following theorem.

THEOREM 3.1 (Weierstrass). *If the series*

$$f_1(z)+f_2(z)+\cdots, \tag{1.35}$$

consisting of functions holomorphic in some region $D \subset C^n$, converges uniformly in that region, then its sum is holomorphic in that region.

Partial derivatives of all orders of $f(z)$ may be obtained by termwise differentiation of the original series. The series thus obtained converge uniformly in the regions indicated above.

Also, the integration of a function $f(z)$ along the curve L (where $\overline{L} \subset D$) or over the surface Q (where $\overline{Q} \subset D$) may be obtained in the case of a bounded region D by termwise integration of the series. An analogous proposition holds as well for n-multiple series

1) Such a function is not to be confused with a biharmonic function satisfying the equation $\Delta\,\Delta u = 0$, where Δ is the Laplace operator.

of holomorphic functions. The proofs of these theorems are carried out in exactly the same way as in the case of one variable. We shall not dwell on them.

Now we indicate a useful generalization of the Weierstrass theorem.

THEOREM 3.1_1. *Suppose that $f(z, \alpha)$ is a holomorphic function of z in some region $D \subset C^n$ for all values of the complex parameter α lying in some neighborhood of α_0, and that the limit*

$$\lim_{\alpha \to \alpha_0} f(z, \alpha) = \phi(z)$$

is approached uniformly in the region D.

Then $\phi(z)$ is a holomorphic function in the region D. In taking derivatives and integrals of it one may carry out these operations under the limit sign.

This theorem is easily obtained from the preceding one if we consider a sequence $\alpha_1, \alpha_2, \cdots$ with $\lim_{n \to \infty} \alpha_n = \alpha_0$. Then

$$\lim_{n \to \infty} f(z, \alpha_n) = \phi(z)$$

is approached uniformly in z, and considering $f(z, \alpha_n)$ as a sum of terms of the series (1.35) we may apply the Weierstrass theorem. Then the assertion of our theorem is easily obtained from the usual formulation of the definition of the limit $\lim_{\alpha \to \alpha_0} f(z, \alpha)$.

We recall that a similar theorem holds in the case of one variable. We used it in the derivation of the Cauchy integral formula in the preceding section.

2. Representation of a holomorphic functional element by power series. First we consider the case of a function of two variables and prove the following fundamental proposition:

THEOREM 3.2. *If the function $f(w, z)$ is holomorphic in the bicylinder*

$$\mathfrak{S} = \mathfrak{S}_{r_1, r_2} = \{ |w - a_1| < r_1, \ |z - a_2| < r_2 \},$$

then at all points of that bicylinder

$$f(w, z) = \sum_{k, l=0}^{\infty} c_{kl} (w - a_1)^k (z - a_2)^l, \tag{1.36}$$

where

$$c_{kl} = \frac{1}{k!\, l!} \left(\frac{\partial^{k+l} f}{\partial w^k \partial z^l} \right)_{\substack{w = a_1 \\ z = a_2}} \tag{1.37}$$

The series (1.36) *converges absolutely and uniformly in the bicylinder* $\mathfrak{S}$. *The representation of the function* $f(w, z)$ *by the series* (1.36) *is unique.*

The first part of this theorem follows from the Cauchy integral formula. For any point (w, z) of the bicylinder $\mathfrak{S}$ we may construct similarly situated bicylinders $\mathfrak{S}_{r_1', r_2'}$ and $\mathfrak{S}_{r_1'', r_2''}$, where $r_k' < r_k'' < r_k$ $(k = 1, 2, \cdots)$ with their centers at (a_1, a_2), containing the point (w, z). If S'' is the skeleton of the boundary of the corresponding bicylinder, then

$$f(w, z) = \frac{1}{(2\pi i)^2} \int_{S''} \frac{f(t_1, t_2)}{(t_1 - w)(t_2 - z)} dt_1 \wedge dt_2. \tag{1.38}$$

Further we have

$$\frac{1}{(t_1 - w)(t_2 - z)} = \sum_{k, l=0}^{\infty} \frac{(w - a_1)^k (z - a_2)^l}{(t_1 - a_1)^{k+1} (t_2 - a_2)^{l+1}}. \tag{1.39}$$

On the basis of the inequality

$$\left| \frac{(w - a_1)^k (z - a_2)^l}{(t_1 - a_1)^{k+1} (t_2 - a_2)^{l+1}} \right| < \frac{1}{r_1' r_2'} \left[\frac{r_1'}{r_1''}\right]^{l+1} \left[\frac{r_2'}{r_2''}\right]^{k+1},$$

which is evident for $(t_1, t_2) \in S''$, this series converges uniformly on S'', and we are justified in integrating it term by term. This last leads us to the expansion (1.36). Therefore one also obtains the equation (1.37).

The fact that the series (1.36) converges uniformly proves the following:

THEOREM 3.3 (Abel). *If at some point* (ω, ζ) *the terms of the power series*

$$\sum_{k, l=0}^{\infty} c_{kl} (w - a_1)^k (z - a_2)^l$$

satisfy the inequalities

$$| c_{kl} (\omega - a_1)^k (\zeta - a_2)^l | < M,$$

then this series converges absolutely and uniformly in the bicylinder $\{ | w - a_1 | < | \omega - a_1 |; \ | z - a_2 | < | \zeta - a_2 | \}$.

If at the point (ω, ζ) *the given series diverges, then it diverges also for* $| w - a_1 | > | \omega - a_1 |, \ | z - a_2 | > | \zeta - a_2 |$.

The proof of this theorem is carried out in exactly the same way as in

classical analysis.

Using the rule for termwise differentiation of a uniformly convergent double series, we easily find that for every other series of the form (1.36) which represents the function $f(w, z)$, the values of the coefficients are expressed by the same equation (1.37). Therefore the uniqueness of the expansion (1.36) follows. Thus Theorem 3.2 is completely proved.

From Abel's theorem it follows as usual that every series of the form

$$\sum_{k, l=0}^{\infty} c_{kl}(w-a_1)^k(z-a_2)^l$$

represents a function holomorphic in each interior point of the region of convergence of that series. Thus we have shown the equivalence of the above definition of a holomorphic function with the definition in the sense of Weierstrass. We may now understand by a function holomorphic at the point (a_1, a_2) the sum of the double power series

$$\sum_{k, l=0}^{\infty} c_{kl}(w-a_1)^k(z-a_2)^l,$$

which converges in some bicylinder

$$\{|w-a_1|<r_1,\ |z-a_2|<r_2\}.$$

We will later refer to the series (1.36) as a double Taylor series.

Note that all functions representing at the point (a_1, a_2) the same functional germ (see subsection 2 of §1) decompose in the neighborhood of this point into the same series (1.36). Therefore we may identify the concepts of holomorphic functional germ at some point (a_1, a_2) and the Taylor series with center at that point.

The double series (1.36) converges absolutely to $f(w, z)$ at all points of $\mathfrak{G}$. Therefore it follows that every ordinary series obtained from (1.36) by arranging its terms in one row will converge to the function $f(w, z)$. Thus one obtains the following theorem.

THEOREM 3.4. *Every function $f(w, z)$ that is holomorphic at the point (a_1, a_2) may be represented in some neighborhood of that point as the sum of a Taylor series*

$$f(w, z)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial}{\partial a_1}(w-a_1)+\frac{\partial}{\partial a_2}(z-a_2)\right]^k f. \tag{1.40}$$

The case of $n > 2$ variables does not introduce any essential complications into our discussion. In what follows we shall speak of a holomorphic functional element at the point $P(a_1, \cdots, a_n)$ or of a function holomorphic at the point $P(a_1, \cdots, a_n)$, meaning by this the sum of the series

$$\sum_{k_1 \cdots k_n = 0}^{n} c_{k_1 \cdots k_n} (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n} = \sum_k c_k (z - a)^k, \tag{1.41}$$

where for brevity we have put $(k_1, \cdots, k_n) = k$, $(z - a)^k = (z_1 - a_1)^{k_1} \cdots$ $\cdots (z_n - a_n)^{k_n}$. As a consequence of the uniqueness of the expansion of a holomorphic function in power series we may regard two holomorphic functional elements at the point z as identical if the values of the coefficients in their expansions coincide. Of course, our power series represents a holomorphic function at all the interior points of the set of points at which it converges. By equation (1.37) this series may be represented as a power series around any other such point.

We shall agree to call the largest polycylinder $\mathfrak{S}_P \{ | z_k - a_k | < R,\ k = 1, \cdots, n \}$ with center at the point P for which the series (1.41) converges an *elementary neighborhood of the point* P for that series, and the radius R of such a polycylinder $\mathfrak{S}_P$ the limiting distance of the point P for the series (1.41). Correspondingly, in considering a holomorphic function in some region D we will call the largest polycylinder $\mathfrak{S}_P \{ | z_k - a_k | < R,\ k = 1, \cdots, n \}$ with center at the point $P(a_1, \cdots, a_n) \in D$ and contained in that region an *elementary neighborhood of the point* P *in the region* D, and the quantity R the *limiting distance of the point* P *in the region* D.

3. n-circular regions. If the function $f(z)$ is holomorphic at the point $P(a)$, then it may be represented in an elementary neighborhood of that point—the polycylinder $\mathfrak{S}_P \{ | z_k - a_k | < R,\ k = 1, \cdots, n \}$—by the power series (1.41). Generally speaking, the set of points of convergence does not exhaust the polycylinder $\mathfrak{S}_P$.

Now we shall establish some general properties of sets of points of convergence of such series. First we shall again consider the case of two variables w, z.

THEOREM 3.5. *The set of points of convergence of the double power series*

$$\sum_{k, l = 0}^{\infty} c_{kl} (w - a_1)^k (z - a_2)^l \tag{1.42}$$

either reduces to the center of the series, the point (a_1, a_2), *or forms some star-shaped region (four-dimensional)* [1] *to which one may adjoin two-dimensional regions of convergence, lying in planes passing through the center of the series parallel to the coordinate planes, and points of convergence situated on the frontier of that region.*

PROOF. Without loss of generality we may place the point (a_1, a_2) at the origin of coordinates, i.e., we put $a_1 = a_2 = 0$. If the series converges at some point (b_1, b_2) not lying in any of the coordinate planes, then by Abel's theorem it converges also in the entire bicylinder $\{|w| < |b_1|, |z| < |b_2|\}$. Suppose that (c_1, c_2), where $c_1, c_2 \neq 0$, is an interior point of that bicylinder. We consider the ray with points whose coordinates are defined by the equations $\tilde{w} = tc_1$, $\tilde{z} = tc_2$, and select the upper bound τ of the numbers t for which at the point $(\tilde{w}, \tilde{z})$ the series (1.42) converges. If there is no such bound, we will take $\tau = \infty$. Then it is evident that the series (1.42) will converge for all t $\{0 \leq t < \tau\}$.

Consider the point $((t_0 + \epsilon) c_1, (t_0 + \epsilon) c_2)$, $0 \leq t_0 < \tau$, at which the series (1.42) converges. Then some neighborhood of the point $(t_0 c_1, t_0 c_2)$ lies in the bicylinder $\{|w| < (t + \epsilon) |c_1|, |z| < (t + \epsilon) |c_2|\}$ in which the series (1.42) converges, and therefore the point $(t_0 c_1, t_0 c_2)$ is always an interior point of the set of points of convergence. The point $(\tau c_1, \tau c_2)$, if $\tau \neq \infty$, lies on the boundary of the region. We note that if $c_1 = 0$, then one can derive from Abel's theorem only the convergence of the series in the disk $\{w = 0, |z| < \tau c_2\}$. In this case it is not possible to say anything directly about the behavior of the series in four-dimensional neighborhoods of that disk.

On each ray $\{z = tc_2\}$ lying in the coordinate plane $w = 0$ we consider the upper bound τ_1 of the numbers t for which 1) the series converges and 2) to which there correspond points having four-dimensional neighborhoods consisting of convergence points for the series (such points in every case lie inside elementary neighborhoods of its center). Then all the points $(0, tc_2)$, where $0 \leq t < \tau_1$, have such neighborhoods. Indeed, if the point $(0, t_0 c_2)$ has such a neighborhood, then the series must converge at some point $(\epsilon_1, (t_0 + \epsilon_2) c_2)$, and thus in the bicylinder $\{|w| < \epsilon_1, |z| < (t_0 + \epsilon_2) c_2\}$ containing neighborhoods of all the points of the ray $\{w = 0, z = tc_2\}$ for $0 \leq t < \tau_1$. Restricting consideration

1) A star-shaped region is any region containing along with any point the entire segment joining that point to the origin of coordinates. Such a region is always homeomorphic to a hypersphere.

only to those points in coordinate planes for which there exist four-dimensional neighborhoods of convergence, we define a certain star-shaped region D, consisting of points of convergence of the series (1.42). In addition to the region itself, the series may turn out to be convergent at some boundary points of that region and also possibly in some two-dimensional regions (evidently these turn out to be annuli) lying in the coordinate planes and adjoining the four-dimensional regions of convergence of D.

REMARK. From Abel's Theorem 3.3 it results that the series (1.42) converges in the region D absolutely and uniformly.

EXAMPLES. 1) The series

$$\sum_{k,\, l=0}^{\infty} w^k z^{l+1} = \frac{z}{(1-w)(1-z)}$$

has as region of convergence D the bicylinder $\{|w| < 1,\ |z| < 1\}$. In the plane $z = 0$ it converges everywhere.

2) The series

$$\sum_{k=0}^{\infty} zw^k = \frac{z}{1-w}$$

has as its region of convergence the region $|w| < 1$. In the plane $z = 0$ it converges everywhere.

A proposition analogous to Theorem 3.5 holds also for n variables. The region $D \subset C^n$ of this theorem is called the *region of uniform convergence* (for short the region of convergence) of the series (1.41). It results from the theorem of Abel that along with each $(z_1^{(0)}, \cdots, z_n^{(0)}) \in D$ there also belong to this region all the points $(z_1, \cdots, z_n)$ for which

$$|z_j - a_j| < |z_j^{(0)} - a_j|, \quad j = 1, \cdots, n.$$

We turn to the study of regions possessing this property.

DEFINITION (of an n-circular region). A region D of the space C^n having the property that along with each point $z^{(0)} \in D$ it contains also all the points

$$[a_1 + (z^{(0)} - a_1)\, e^{i\theta_1}, \cdots, a_n + (z_n^{(0)} - a_n)\, e^{i\theta_n}] \in D,$$

where $\theta_1, \cdots, \theta_n$ are any real numbers, satisfying the condition $0 \le \theta_j \le 2\pi$, is called an n-circular region with center at the point $(a_1, \cdots, a_n)$.

Thus an n-circular region is mapped onto itself by transformations of the

group

$$z_j = (z_j^{(0)} - a_j)\, e^{i\theta_j} + a_j, \quad 0 \leq \theta_j \leq 2\pi, \quad j = 1, \cdots, n. \tag{1.43}$$

One says that equations (1.44) define the *group of automorphisms* of the n-circular region D.

If along with each point $z^{(0)}$ there belong to the region D all the points z for which

$$|z_j - a_j| \leq |z_j^{(0)} - a_j|, \quad j = 1, \cdots, n,$$

the region D is said to be a *complete* n-circular region. We have already noted that *the region of convergence of the power series* (1.41) *has from Abel's theorem this property and accordingly is a complete n-circular region.* One may show that every function holomorphic in some n-circular region D containing its center may be represented in it by the power series (1.41). However, generally speaking, such a series will converge up to the limits of that region. It will have as its region of convergence some complete n-circular region containing the region D. One may further show that not every complete n-circular region is such a region of convergence. The special properties of complete n-circular regions which are regions of convergence of the power series (1.41) will be considered below. From each plane

$$z_1 = \text{const}, \cdots, z_{j-1} = \text{const}, \quad z_{j+1} = \text{const}, \cdots, z_n = \text{const},$$

with which a complete n-circular region intersects, it excises a complete disk. We note further that a complete n-circular region is always star-shaped.

In order to characterize an n-circular region D with center at zero it is convenient to consider the so-called "absolute octant" R_n^+ of the space R_n in which the coordinates are the moduli $|z_1|, \cdots, |z_n|$. In this octant there corresponds to the region D its *image*, namely, an open set D^+ which completely defines it. If the region D is complete, then its image D^+ is also a region. In this case along with each point $(|z_1^{(0)}|, \cdots, |z_n^{(0)}|) \in D^+$ there also belongs to this octant the entire prism consisting of the points $(|z_1|, \cdots, |z_n|)$ satisfying the condition

$$|z_j| \leq |z_j^{(0)}|, \quad j = 1, \cdots, n.$$

In the consideration of a bicircular region D with center at zero of the space C^2 of complex variables w and z the role of the absolute octant is played by the "absolute quarter-space" in which the quantities $|w|$ and $|z|$ serve as

coordinates.

DEFINITION (of associated radii of convergence). The numbers $r_1, \cdots, r_n > 0$ are said to be associated radii of convergence of the power series (1.41) if that series converges in the polycylinder $S\{|z_j - a_j| < r_j,\ j = 1, \cdots, n\}$ and diverges for $|z_j - a_j| > r_j$ $(j = 1, \cdots, n)$.

The existence of associated radii of convergence follows from the completeness of an n-circular region which is the region of convergence of the series (1.41).

From the definition of associated radii of convergence it follows that there exists a functional dependence $\psi(r_1, \cdots, r_n) = 0$. Putting in this relation $r_j = |z_j - a_j|$ $(j = 1, \cdots, n)$, we obtain an equation $\psi(|z_1 - a_1|, \cdots, |z_n - a_n|) = 0$, defining the boundary of the region D of convergence of the series (1.41). In the octant R_n^+ this equation (for $a_j = 0$) defines a hypersurface which, along with the corresponding portions of the coordinate hyperplanes, bounds the image D^+ of the region D.

The upper limits of the values r_j will be denoted by R_j and called the *maximal radii of convergence* of the series (1.41). Evidently the polycylinder $\{|z_j - a_j| < R_j,\ j = 1, \cdots, n\}$ is the smallest containing an n-circular region of convergence of the series (1.41).

Now we suppose that $n = 2$. From the definition of associated radii of convergence it follows that the function $r_2 = \phi(r_1)$ may be constant on separate portions of the interval $0 < r_1 < R_1$. To some values of r_1 of this interval there may correspond entire segments of the interval $0 < r_2 < R_2$. Consequently the curve $r_2 = \phi(r_1)$ may contain segments of lines parallel to the coordinate axes.

From the definition of associated radii of convergence it further results that if r_1, r_2 and r_1', r_2' are two pairs of associated radii of convergence and $r_1 \leq r_1'$, then $r_2 \geq r_2'$ (monotonicity property of the function $r_2 = \phi(r_1)$).

The following theorem is valid:

THEOREM 3.6. *If r_1 and r_2 are associated radii of convergence of the double power series* (1.42), *then the sum of this series cannot be holomorphic at all the points of the surface* $\{|w - a_1| = r_1,\ |z - a_2| = r_2\}$.

PROOF. We take $a_1 = a_2 = 0$, which does not affect the generality of our discussion. From the hypotheses of Theorem 3.6 it follows that the sum of the series (1.42) is a function $f(w, z)$ holomorphic in the bicylinder

$\mathfrak{S}\{|w| < r_1, |z| < r_2\}$. This function cannot be holomorphic at all points of the boundary $\partial\mathfrak{S}$ of the bicylinder $\mathfrak{S}$. In the contrary case, since the boundary $\partial\mathfrak{S}$ is closed, the function $f(w, z)$ would be holomorphic in some bicylinder $\mathfrak{S}_1\{|w| < R_1, |z| < R_2\}$, where $R_1 > r_1$, $R_2 > r_2$, and the numbers r_1 and r_2 would not be associated radii of convergence of the series (1.42).

Suppose that in spite of the assertion of the theorem the function $f(w, z)$ is holomorphic at all points of the surface $\{|w| = r_1, |z| = r_2\}$. Then from what has been said it cannot be holomorphic at all points of the hypersurface $\{|w| = r_1, |z| < r_2\}$, $\{|w| < r_1, |z| = r_2\}$. We shall show that under our assumptions the function $f(w, z)$ is holomorphic on these hypersurfaces.

As a consequence of the fact that the surface $\{|w| = r_1, |z| = r_2\}$ is closed, there exist in our case two numbers ϵ and δ such that the function $f(w, z)$ is also holomorphic in all the points of the region G defined by the inequalities $|w - re^{i\theta}| < \epsilon$, $|z - r_2e^{i\phi}| < \delta$ for some ϕ and θ. Suppose that (w_0, z_0) is any interior point of the bicylinder $\mathfrak{S}$, where $|z_0| \le r_0 - \delta$. Then the disk

$$\{w = w_0, |z| \le r_2 - \tfrac{1}{2}\delta\}$$

lies inside the bicylinder $\mathfrak{S}$, and therefore we may write

$$f(w_0, z_0) = \frac{1}{2\pi i}\int_{|\zeta| = r_2 - \delta/2} \frac{f(w_0, \zeta)}{\zeta - z_0}\, d\zeta.$$

But the function $f(w, \zeta)$ is holomorphic in the disk $\{|w| \le r_1 - \tfrac{1}{2}\epsilon,\ z = \zeta\}$ (since that disk lies in the bicylinder $\mathfrak{S}$) and in the annulus $\{z = \zeta,\ r_1 - \tfrac{1}{2}\epsilon \le |w| \le r_1 + \tfrac{1}{2}\epsilon\}$ (since that annulus lies in G). It therefore will be holomorphic in the disk $\{z = \zeta, |w| \le r_1 + \tfrac{1}{2}\epsilon\}$. Then in view of the Cauchy integral formula

$$f(w_0, \zeta) = \frac{1}{2\pi i}\int_{|\tau| = r_1 + \epsilon/2} \frac{f(\tau, \zeta)}{w_0 - \tau}\, d\tau,$$

we may also write

$$f(w_0, z_0) = -\frac{1}{4\pi^2}\iint_{\substack{|\zeta| = r_2 - \delta/2 \\ |\omega| = r_1 + \epsilon/2}} \frac{f(\omega, \zeta)}{(\omega - w_0)(\zeta - z_0)}\, d\omega \wedge d\zeta.$$

This means that the function $f(w, z)$ is holomorphic in the bicylinder $\{|w| < r_1 + \tfrac{1}{2}\epsilon,\ |z| < r_2 - \tfrac{1}{2}\delta\}$ and accordingly, since δ is arbitrary, is

holomorphic on the hypersurface $\{|w| = r_1, |z| < r_2\}$. In exactly the same way we can show that this function is also holomorphic on the hypersurfaces $\{|w| < r_1, |z| = r_2\}$. Therefore, as we have seen, our assertion follows.

It follows easily from Theorem 3.6 that the sum of a double power series cannot be holomorphic at all points of the boundary of an elementary neighborhood of the center of the series. Such a neighborhood $\{|w-a_1| < R, |z-a_2| < R\}$ corresponds to equal values of the associated radii of convergence. To the skeleton of that neighborhood there corresponds (for $a_1 = a_2 = 0$) on the absolute quarter-plane the point of intersection of the bisector of the coordinate angle with the boundary of the region of convergence of the series (1.42).

A theorem analogous to Theorem 3.6 holds also in the general case of n variables.

THEOREM 3.7. *The associated radii of convergence $r_1, r_2, \cdots, r_n$ of the power series* (1.41) *satisfy the relation*

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k| r^k} = 1.$$

Here and in what follows $r^k = r_1^{k_1} \cdots r_n^{k_n}$, $\|k\| = k_1 + \cdots + k_n$.

PROOF. This is a generalization of the classical formula of Cauchy-Hadamard. It will be carried through for the adjoint radii of convergence r and r' of the double power series $\sum_{k,l=0}^{\infty} c_{kl} w^k z^l$. We consider points with coordinates $w = rt$, $z = r't$ $(t > 0)$. From the definition of associated radii of convergence it follows that the series

$$\sum_{k,l=0}^{\infty} |c_{kl}| w^k z^l = \sum_{m=0}^{\infty} \sum_{j=0}^{m} |c_{j,m-j}| r^j r'^{m-j} t^m$$

converges for $t < 1$ and diverges for $t > 1$. Therefore

$$\overline{\lim_{m\to\infty}} \sqrt[m]{\sum_{j=0}^{m} |c_{j,m-j}| r^j r'^{m-j}} = 1. \tag{1.44}$$

Denote by j_m that index j for which

$$|c_{j_m, m-j_m}| r^{j_m} r'^{m-j_m} = \max_{0 \le j \le m} |c_{j,m-j}| r^j r'^{m-j}.$$

Then

$$|c_{j_m, m-j_m}| r^{j_m} r'^{m-j_m} \leq \sum_{j=0}^{m} |c_{j, m-j}| r^j r'^{m-j} \leq$$
$$\leq m |c_{j_m, m-j_m}| r^{j_m} r'^{m-j_m}$$

Hence using relation (1.44) we find that

$$\overline{\lim_{k+l \to \infty}} \sqrt[k+l]{|c_{kl}| r^k r'^l} = 1.$$

4. **Estimates of the Taylor coefficients.** First of all we note that for the coefficients of the series (1.41), as in the case of one variable, the *Cauchy inequalities* hold. This means that if in the polycylinder $S\{|z_j - a_j| < r_j,$ $j = 1, \cdots, n\}$ the function $f(z)$ is holomorphic, satisfies the condition $|f(z)| < M$ and is represented by the series (1.41), then

$$|c_k| \leq \frac{M}{r^k}, \tag{1.45}$$

where again $k = (k_1 \cdots k_n)$, $r^k = r_1^{k_1} \cdots r_n^{k_n}$.

In order to simplify the notation we place the center of the series in question and of the n-circular regions at the origin of coordinates. Further, for the bounded n-circular region D with center at the origin of coordinates we put

$$d_k(D) = \sup_{z \in D} |z|^k = \sup_{|z| \in D^+} |z|^k.$$

Then the following theorem [1)] holds:

THEOREM 3.8. *If the function $f(z)$ is holomorphic in the closed bounded complete n-circular region D and is represented in it by the series* (1.41), *then*

$$|c_k| \leq \frac{\max_{z \in \overline{D}} |f(z)|}{d_k(D)}.$$

PROOF. For any polycylinder $S_{|\zeta|} = \{|z_j| < |\zeta_j|, \ j = 1, \cdots, n\} \subset D$ the Cauchy inequalities (1.45) yield

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1) Theorems 3.8, 3.9, and 3.10 are due to L. A. Aĭzenberg and B. S. Mitjagin [1]. The proof of Theorem 3.9 presented in what follows is due to L. I. Ronkin. In a special case Theorem 3.8 was obtained earlier by A. A. Temljakov. Estimates of the Taylor coefficients for various classes of functions will be found in the papers of I. I. Bavrin, for example, in his paper [1].

$$|c_k| \le \frac{\max_{z \in \overline{D}} |f(z)|}{|\zeta|^k}.$$

Substituting in the right side of this inequality the lower bound of the quantities appearing there over all polycylinders with center at the origin of coordinates and lying in the region D, we obtain the inequalities which express Theorem 3.8.

5. Some properties of a holomorphic functional element.

THEOREM 3.9. *If the series* (1.41) (*where* $a_j = 0$) *converges in a bounded complete* n*-circular region* D, *then for each* n*-circular region* D_0, $\overline{D}_0 \subset D$, *the following series converges:*

$$\sum_k |c_k|\, d_k(D_0).$$

PROOF. Suppose that the region D_r consists of points $z = (z_1, \cdots, z_n)$ for which the points $(z_1/r, \cdots, z_n/r) \in D$. Here $0 < r < \infty$. We choose the numbers r_1 and r_2, where $0 < r_1 < r_2 < 1$, such that $\overline{D}_0 \subset D_{r_1} \subset D_{r_2} \subset D$. Then the sum of the series (1.41) is bounded in modulus in the region D_{r_2} by some constant M. Taking into account the relations

$$d_k(D_0) \le d_k(D_{r_1}) = r_1^{\|k\|}\, d_k(D); \quad d_k(D_{r_2}) = r_2^{\|k\|}\, d_k(D),$$

we obtain

$$\sum_k |c_k|\, d_k(D_0) \le \sum_k |c_k|\, d_k(D_{r_1}) \le M \sum_k \frac{d_k(D_{r_1})}{d_k(D_{r_2})} =$$

$$= M \sum_k \left(\frac{r_1}{r_2}\right)^{\|k\|} < \infty.$$

THEOREM 3.10. *In order that the series* (1.41) *should converge* (*with* $a_j = 0$) *in the bounded complete* n*-circular region* D, *it is necessary and sufficient that the series*

$$\sum_k c_k\, d_k(D)\, z^k \tag{1.46}$$

should converge in the unit polycylinder E. *Here and in what follows*

$$z^k = z_1^{k_1} \cdots z_n^{k_n}.$$

PROOF. Sufficiency. Consider the region D_r. Suppose that the series

(1.46) converges in the polycylinder E. Then the series

$$\sum_k |c_k|\, d_k(D)\, r^{\|k\|} \tag{1.47}$$

converges for any $r < 1$. But since $d_k(D_r) = d_k(D)\, r^{\|k\|}$, the series (1.41) converges in the region D_r for any $r < 1$. This means that the series (1.41) converges in the region D.

Necessity. Suppose that the series (1.41) converges in the region D. Then from Theorem 3.9 the series $\Sigma_k\, |c_k|\, d_k(D_r)$ converges for any $r < 1$. Therefore it follows that the series (1.47) also converges for any $r < 1$. Therefore the series (1.46) converges in the polycylinder E.

From Theorems 3.7 and 3.10 it follows that the following corollary holds.

COROLLARY. *For the convergence of the series* (1.41) *in the region D it is necessary and sufficient that*

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k|\, d_k(D)} \le 1.$$

The expansion of a holomorphic functional element in Taylor series makes it possible to prove a series of important propositions. We shall now dwell on a number of them.

THEOREM 3.11. *If the function $f(z)$ is holomorphic in the region $D \subset C^n$ and is not constant there, then $|f(z)|$ cannot take on its maximum value inside the region. If the function $|f(z)|$ is also continuous in the closed region $\overline{D}$, then it takes on its maximum value on the boundary ∂D of the region D.*

Indeed, if it happens that $|f|$ takes on its maximum value at a point $P(a)$ of the region D, then we choose a neighborhood of the point P defined by the conditions $\{|z_k - a_k| < r,\ k = 1, \cdots, n\}$, where r is chosen so that the neighborhood just defined belongs to the region D. Then the functions of one variable $f(z_1, a_2, \cdots, a_n)$, $f(a_1, z_2, \cdots, a_n), \cdots$ are holomorphic in the disks $|z_1 - a_1| < r_1$, $|z_2 - a_2| < r_2, \cdots$, and they take on their maxima at the centers of these disks. But this can happen only in the case when $f(z)$ is constant in all the variables $z_1, \cdots, z_n$ in that neighborhood of the point P. This last follows from equations (1.36) and (1.37).[1)] If $Q(b)$ is any other point of the region

1) Since for this it is necessary that all the coefficients of the series (1.36) vanish.

D, we join the points P and Q by some curve lying in the region D and cover this curve by neighborhoods (of the sort just discussed) of a finite number of its points. We choose the neighborhoods so that each point lies in the neighborhood of the preceding one. Using what has already been proved we easily find that the function $f(z)$ is constant in all the neighborhoods thus chosen and that accordingly

$$f(b_1, \cdots, b_n) = f(a_1, \cdots, a_n),$$

i.e., our function is constant over the entire region.

The last assertion of the theorem follows from the Weierstrass theorem on the maximum of a continuous function.

THEOREM 3.12 (Liouville). *If the function $f(z)$ is holomorphic at all the points of the space C^n and its modulus is everywhere bounded by one and the same number, then that function is constant.*

Consider first the case of two variables. Suppose that (a_1, a_2) and (b_1, b_2) are two arbitrary points. Consider the functions $f(w, b_2)$ and $f(a_1, z)$. Under the conditions of the theorem, it follows from the Liouville theorem for the case of one variable that these functions are constants. Accordingly, $f(b_1, b_2) = f(a_1, b_2)$; $f(a_1, a_2) = f(a_1, b_2)$, so that $f(b_1, b_2) = f(a_1, a_2)$. Thus our theorem is true for $n = 2$. For the case of three variables z_1, z_2, z_3 we consider two arbitrary points (a_1, a_2, a_3) and (b_1, b_2, b_3) and take the functions $f(z_1, z_2, b_3)$, $f(a_1, z_2, z_3)$.

By what has been proved they are constant at all points of the spaces z_1, z_2 and z_2, z_3. It therefore follows that $f(b_1, b_2, b_3) = f(a_1, a_2, b_3)$ (we again put $z_1 = a_1$, $z_2 = a_2$) and $f(a_1, a_2, a_3) = f(a_1, a_2, b_3)$ (we again put $z_2 = a_2$, $z_3 = b_3$). Thus, $f(b_1, b_2, b_3) = f(a_1, a_2, a_3)$.

Continuing the discussion in an analogous way we see that our assertion is valid for any number of variables.

From the expression (1.41) there further follows the so-called *uniqueness theorem.*

THEOREM 3.13. *If the function $f(z)$ is holomorphic in the region D and if at some point a of the region D this function and all of its partial derivatives vanish, then in the region D the function $f(z)$ is identically zero.*

PROOF. From (1.41) it follows that in some polycylinder $\{|z_k - a_k| < r, k = 1, \cdots, n\}$ the values of this function are equal to zero. We join an arbitrary

interior point of the region not lying in this polycylinder with the point a by a curve L lying entirely in the region D. Suppose that c is the point of the curve L closest to the point a in any neighborhood of which we have $f(z) \not\equiv 0$. Then in a neighborhood of it one may find a point b such that at that point the function $f(z)$ and all of its partial derivatives vanish, and in addition the point c lies in the polycylinder $\{|z_k - b_k| < r', k = 1, \cdots, n\}$, lying along with its boundary in the region D. Then, in spite of our supposition, there exists a neighborhood of the point c in which $f(z) \equiv 0$. The theorem is proved.

From the theorem just proved there follows also: [1]

THEOREM 3.14. *If in the region D the functions $f(a)$ and $\phi(z)$ are holomorphic and their values and those of all of their partial derivatives coincide with each other at some point P of that region, then the functions f and ϕ coincide with each other at all the points of D.*

We note that conversely, in order that all the partial derivatives of the function $f(z)$ at P should vanish, it is not necessary to require that the function $f(z)$ should be zero throughout a neighborhood of the point P. It suffices to require this only at the points of some series of sequences $P_n^{(k)}(\lim_{n\to\infty} P_n^{(k)} = P)$, such that one may obtain in the form

$$\lim_{n\to\infty} \frac{f(P_n^{(k)}) - f(P)}{r(P_n^{(k)}, P)}, \cdots \quad (k = 1, 2, \cdots)$$

all the partial derivatives of any order (or linear combinations of the latter, whose vanishing would imply that all the partial derivatives are zero). Here $r(P_n^{(k)}, P)$ is the distance between the points P, $P_n^{(k)}$.

6. **Theorems on integrals of holomorphic functions.** In what follows it will be useful to make use of some propositions closely related to the Weierstrass theorem on functional series.

THEOREM 3.15. *Suppose that $f(w, z, \alpha)$ is a continuous function of the variables w, z, α, where the point $P(w, z)$ lies in some region D of the space C^2 of the variables w, z, and the point α is on a piecewise smooth curve L in the plane of the variable α (the endpoints of L included). Suppose further that for all $\alpha \in L$ the function $f(w, z, \alpha)$ is a holomorphic function of w, z in the*

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1) Other uniqueness theorems may be found in the papers of Aĭzenberg [5] and Carafa [1].

region D. Then $\phi(w, z) = \int_L f(w, z, \alpha)\, d\alpha$ is a holomorphic function of w, z in the region D.

We shall show that the following integral exists:

$$\phi'_w = \int_L f'_w(w, z, \alpha)\, d\alpha$$

(for ϕ'_z analogously). To this end we estimate the quantity

$$\frac{\phi(w + \Delta w, z) - \phi(w, z)}{\Delta w} - \int_L f'_w(w, z, \alpha)\, d\alpha =$$

$$= \int_L \left[\frac{f(w + \Delta w, z, \alpha) - f(w, z, \alpha)}{\Delta w} - f'_w(w, z, \alpha) \right] d\alpha =$$

$$= \frac{\Delta w}{2\pi i} \int_L d\alpha \int_{C_r} \frac{f(\omega, z, \alpha)}{(\omega - w - \Delta w)^2 (\omega - w)}\, d\omega.$$

The last expression is obtained by using the Taylor formula with remainder term after the first derivative from the theory of analytic functions of one variable. Here C_r is the circle $\{|w - \omega| = r\}$. Its radius r is taken so that $f(\omega, z, \alpha)$, as a function of ω, for fixed z, α, is holomorphic in the disk $|w - \omega| < r$. It is obvious that as $\Delta w \to 0$ the double integral thus obtained remains bounded, from which the required result follows.

THEOREM 3.16. *If the function $f(w, z)$ is holomorphic in the bicylinder*

$$E\{|w - w_0| < r, \ |z - z_0| < r\},$$

and L_1, L_2 are piecewise smooth curves lying in the region E such that

$$L_1 \subset \{|w - w_0| < r, \ z = \text{const}\},$$

$$L_2 \subset \{|z - z_0| < r, \ w = \text{const}\},$$

then the functions

$$F(w, z) = \int_{w_0 L_1}^{w} f(\omega, z)\, d\omega; \quad \Phi(w, z) = \int_{z_0 L_2}^{z} f(w, \zeta)\, d\zeta$$

are holomorphic in the bicylinder E, while

$$F'_w = f(w, z); \quad \Phi'_z = f(w, z).$$

PROOF. That the function $F(w, z)$ is holomorphic in z and the function

$\Phi(w, z)$ in w follows from the theorem just proved. If we move along the curve L_1 we do not change the value of z, and w does not change along the curve L_2. We may therefore apply to the functions $F(w, z)$, $\Phi(w, z)$ the corresponding theorem of the integral calculus from the theory of functions of one complex variable. On this basis we conclude that $F(w, z)$ is a holomorphic function of w, that $F'_w = f$, and that $\Phi(w, z)$ is a holomorphic function of z, $\Phi'_z = f$.

Thus the functions $F(w, z)$ and $\Phi(w, z)$ are holomorphic in the region E in each variable separately. Therefore our assertion follows.

§4. PREPARATION THEOREM OF WEIERSTRASS. ANALYTIC SETS AND SURFACES

1. **Weierstrass' preparation theorem.** If the function $f(z)$ of one variable is holomorphic at the point z_0 and $f(z_0) = a$, then in the neighborhood of this point $f(z) = a + (z - z_0)^m \phi(z)$, where m is an integer, the function $\phi(z)$ is holomorphic at the point z_0, while $\phi(z_0) \neq 0$. For the case $z_0 = a = 0$ this representation has the form: $f(z) = z^m \phi(z)$, where $\phi(0) \neq 0$.

The corresponding representation of a holomorphic function of two variables in the neighborhood of a zero, which without loss of generality we may take to be situated at the origin of coordinates, is given by the following theorem.

THEOREM 4.1 (Preparation theorem of Weierstrass for functions of two variables). *If the function $F(w, z)$ is holomorphic at the point $(0, 0)$, while $F(0, 0) = 0$ and $F(w, z) \not\equiv 0$, then in some bicylinder $\{|w| < r,\ |z| < h\}$*

$$F(w, z) = z^{\mu}[w^m + A_1(z)w^{m-1} + \cdots + A_m(z)]\,\Omega(w, z). \tag{1.48}$$

Here the integers $m, \mu \geq 0$, while the functions $A_k(z)$, $\Omega(w, z)$ are holomorphic at the origin of coordinates, and $A_k(0) = 0$, $k = 1, \cdots, m$; $\Omega(0, 0) \neq 0$. The functions $A_k(z)$ and $\Omega(w, z)$ are uniquely defined by the conditions of the theorem.

PROOF. We consider the function $F(w, 0)$. If $F(w, 0) \not\equiv 0$, then we may deduce (1.48), taking $\mu = 0$ there. If $F(w, 0) \equiv 0$, then, operating with $F(w, z)$ as with a function of the single variable z, we represent it in some disk $|z| < \eta$, for all w satisfying the condition $|w| < \rho$, in the form

$$F(w, z) = z^{\mu} F_1(w, z). \tag{1.49}$$

Here the integer $\mu > 0$, $F_1(w, 0) \not\equiv 0$. Thus, in both cases it is sufficient for us

to consider a function $F(w, z)$ for which $F(w, 0) \not\equiv 0$ and to prove that in some bicylinder $\{|w| < r, |z| < h\}$,

$$F(w, z) = [w^m + A_1(z) w^{m-1} + \cdots + A_m(z)]\Omega(w, z). \tag{1.50}$$

For the case when $F(w, 0) \equiv 0$ we obtain the representation (1.48) by using (1.49) and equation (1.50) for $F_1(w, z)$. Thus, suppose that the function $F(w, z)$ $(F(w, 0) \not\equiv 0)$ is holomorphic in the bicylinder $\{|w| < r_1, |z| < h_1\}$. We are given that the function $F(w, 0)$ vanishes for $w = 0$. Suppose that m is the order of this zero. Then we can always find a disk $|w| \leq r$ $(0 < r < r_1)$ such that in this disk $w = 0$ is the only zero of the function $F(w, 0)$. We may further choose $h \leq h_1$ so that $F(w, z) \neq 0$ for $|w| = r$, $|z| < h$. From the theorem on the logarithmic residue it follows that the number of zeros of the function $F(w, z_0)$ (where $|z_0| < h$) in the disk $|w| < r$ is determined by the value of the integral

$$\frac{1}{2\pi i} \int_{|w|=r} \frac{F'_w(w, z_0)}{F(w, z_0)}\, dw = m. \tag{1.51}$$

(This integral is equal to m for $z_0 = 0$. It is continuous for $|z_0| < h$ and must always be equal to an integer. Therefore (1.51) follows for all z_0 satisfying the condition $|z_0| < h$.) We denote by $w_k(z)$, $k = 1, \cdots, m$, the roots of the function $F(w, z)$ corresponding to some value z of the disk $|z| < h$. Further we make use of the fact that if $\phi(w)$ is some function holomorphic in the disk $|w| < r$ and continuous in the closed disk $|w| \leq r$, then (as one obtains from the Cauchy residue theorem)

$$\sum_{k=1}^{m} \phi(w_k) = \frac{1}{2\pi i} \int_{|w|=r} \phi(w) \frac{F'_w(w, z)}{F(w, z)}\, dw. \tag{1.52}$$

Putting successively $\phi(w) = w^s$, where $s = 1, \cdots, m$, we form the functions $Q_s(z) = \Sigma_{k=1}^m w_k^s$. Further, by the usual formulas of algebra, we may express in terms of the $Q_s(z)$ the coefficients $A_l(z)$ of the polynomial $w^m + A_1(z) w^{m-1} + \cdots$ $\cdots + A_m(z)$, having $w_k(z)$ as its roots. For $z = 0$ all the $w_k = 0$ and therefore all the $A_l(0) = 0$. Evidently this polynomial is defined, under our hypotheses, in a unique way.

Consider the function $\Omega(w, z)$ equal to the quotient on division of $F(w, z)$ by this polynomial. For each z of the disk $|z| < h$ it is a holomorphic function of w everywhere in the disk $|w| \leq r$, except for the points w_k. At the latter,

from the nature of its definition, it has removable singularities. We extend the function Ω to these points by putting its values equal to the corresponding limit values. Then in the disk $|w| < r$ this function is represented by the Cauchy integral

$$\Omega(w, z) = \frac{1}{2\pi i} \int_{|W|=r} \Omega(W, z) \frac{dW}{W - w}. \tag{1.53}$$

This integral, from Theorem 3.16 (we must recall that for $|w| = r$, $|z| < h$ the function $F(w, z) \neq 0$, and therefore $\Omega(w, z)$ is a holomorphic function of z in the disk $|z| < h$ for $|w| = r$), defines a holomorphic function of w, z in the entire bicylinder $\{|w| < r, |z| < h\}$. Evidently $\Omega(w, z) \neq 0$ in this bicylinder. Hence we obtain the representation (1.48) for the function $F(w, z)$.

The preparation theorem of Weierstrass is valid for holomorphic functions of any number of variables.

THEOREM 4.2 (Preparation theorem of Weierstrass for functions of $n + 1$ variables). *If $F(w, z_1, \cdots, z_n)$ is a holomorphic function at the origin of coordinates, while $F(0, 0, \cdots, 0) = 0$, $F(w, 0, \cdots, 0) \not\equiv 0$, then in some neighborhood of the origin of coordinates*

$$F(w, z_1, \cdots, z_n) = (w^m + A_1 w^{m-1} + \cdots + A_m)\, \Omega(w, z_1, \cdots, z_n). \tag{1.54}$$

Here $A_k(z_1, \cdots, z_n)$, $\Omega(w, z_1, \cdots, z_n)$, $k = 1, \cdots, m$ are functions holomorphic at the origin of coordinates, while $A_k(0, \cdots, 0) = 0$, $\Omega(0, \cdots, 0) \neq 0$; m is the order of the zero of the function $F(w, 0, \cdots, 0)$ at the point $w = 0$.

The functions A_k and Ω are uniquely defined by the conditions of the theorem.

The proof of this proposition is carried out verbatim as in the proof of the preparation theorem of Weierstrass for two variables.

In the case when $F(w, 0, \cdots, 0) \equiv 0$, a direct generalization of equation (1.48) turns out to be impossible, as may be shown by appropriate examples. 1) However, one may prove the following theorem.

THEOREM 4.3. *If the function $F(z) \not\equiv 0$ of the complex variables z_0, $z_1, \cdots, z_n$ is holomorphic at the point $z = 0$, then as a result of a substitution of the form*

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1) See Osgood [1], p. 90.

$$z_k = \sum_{s=0}^{n} a_{ks} w_s, \quad k = 0, 1, \cdots, n, \quad \operatorname{Det} a_{ks} \neq 0,$$

one has $F(z_0, z_1, \cdots, z_n) = \Phi(w_0, w_1, \cdots, w_n)$, *where* $\Phi(w_0, 0, \cdots, 0) \not\equiv 0$. *Then the function* Φ *may be represented in the form* (1.54).

PROOF. For conciseness of notation we consider the case of two variables. Expand the function $F(z_0, z_1)$ in a Taylor series in the neighborhood of the origin. Suppose that this series begins with non-zero terms of degree m:

$$F(z_0, z_1) = c_0 z_0^m + c_1 z_0^{m-1} z_1 + \cdots + c_m z_1^m + \cdots. \tag{1.55}$$

Here not all the coefficients c_k are equal to zero.

Consider the coefficient of w_0^m for the function $\Phi(w_0, w_1)$. It is easy to calculate that it is equal to $c_0 a_{00}^m + c_1 a_{00}^{m-1} a_{10} + \cdots + c_m a_{10}^m$. Evidently under our hypotheses the quantities a_{00}, a_{10} may always be so chosen that this expression is not equal to zero for $\operatorname{Det} a_{ks} \neq 0$. Thus our assertion is proved.

A function $F(z_0, \cdots, z_n)$, holomorphic at the point $z = 0$, for which $F(0, \cdots, 0) = 0$, but $F(z_0, 0, \cdots, 0) \not\equiv 0$, will be called *regular* at the point $z = 0$ in the variable z_0. Analogously one introduces the concept of a function regular at any point $z = a$ relative to some variable z_k.

2. **The ring of functions holomorphic at some point.** The functions of the complex variables $z_1, \cdots, z_n$, holomorphic at some point $a(a_1, \cdots, a_n)$, evidently form a commutative ring $\mathfrak{O}_a^{(n)}$ with identity. This follows from the fact that the sum, difference and product of two such functions, formed by the usual rules, are again functions holomorphic at that point. The ring $\mathfrak{O}_a^{(n)}$ is frequently also called the ring of convergent power series with center at the point $z = a$. It does not contain divisors of zero (the product of two holomorphic functions is equal to zero only in the case when one of the factors is equal to zero) and therefore is an integral domain ("ring of integrity").

The ring $\mathfrak{O}_a^{(n)}$ contains divisors of unity. These are *invertible* functions $f \in \mathfrak{O}_a^{(n)}$, for which also $f^{-1} \in \mathfrak{O}_a^{(n)}$. Evidently these functions f are represented by power series with center at the point $z = a$ with a free term distinct from zero. Two functions $\psi, \phi \in \mathfrak{O}_a^{(n)}$ will be called *equivalent* to one another if in some neighborhood of the point a we have $\psi = \phi\eta$, where the function $\eta \in \mathfrak{O}_a^{(n)}$ is also a divisor of unity.

A function $p \in \mathfrak{O}_a^{(n)}$ which in the neighborhood of the point $z = a$ may be represented in the form

$$p(z_1, \cdots, z_n) = A_0(z_n - a_n)^m + A_1(z_n - a_n)^{m-1} + \cdots + A_m, \tag{1.56}$$

where the functions $A_k \in \mathfrak{O}_a^{(n-1)}$ ($\mathfrak{O}_a^{(n-1)}$ the ring of functions of the variables $z_1, \cdots, z_{n-1}$ holomorphic at the point $z = a$), is called a *pseudopolynomial* with center at the point $(a_1, \cdots, a_{n-1})$. The ring of these pseudopolynomials we will denote in what follows by the symbol $\mathfrak{O}_a^{(n-1)}[z_n - a_n]$. A pseudopolynomial p for which the coefficient A_0 is a divisor of unity is said to be *distinguished* or *normalized*. The Preparation Theorem of Weierstrass 4.2 demonstrates that each function $F \in \mathfrak{O}_a^{(n)}$ which is regular in the variable z_n at the point $z = a$ is equivalent to some distinguished pseudopolynomial of the ring $\mathfrak{O}_a^{(n-1)}[z_n - a_n]$.

By its definition a pseudopolynomial is a polynomial in powers of $(z_n - a_n)$ with coefficients which are holomorphic functions of the variables $z_1, \cdots, z_{n-1}$ in some neighborhood of the point $z = a$. Its roots are continuous functions of these coefficients and accordingly depend continuously on the variables $z_1, \cdots \cdots, z_{n-1}$. Thus, from the preparation theorem of Weierstrass it follows, in distinction from the case of one variable, that in any neighborhood of each zero of a holomorphic function of $n > 1$ variables there is an infinite set of other zeros of it.

If for the functions $p, q, r \in \mathfrak{O}_a^{(n)}$, in some neighborhood of the point a the equation $p = qr$ holds, while neither the function q nor the function r is a divisor of unity, then the function p is said to be *reducible* in the ring $\mathfrak{O}_a^{(n)}$ (or at the point $z = a$). If the function p can never be represented in the indicated form, then it is said to be *irreducible* in the ring $\mathfrak{O}_a^{(n)}$ (or at the point $z = a$).

THEOREM 4.4. *A distinguished pseudopolynomial $p \in \mathfrak{O}_a^{(n-1)}[z_n - a_n]$, reducible in the ring $\mathfrak{O}_a^{(n)}$, is reducible in the ring $\mathfrak{O}_a^{(n-1)}[z_n - a_n]$. The factors into which it decomposes are distinguished pseudopolynomials of that ring.*

PROOF. If in some neighborhood of the point a we have $p = qr$, while $q, r \in \mathfrak{O}_a^{(n)}$, then the functions q and r are evidently regular in the variable z_n at the point $z = a$. Therefore in some neighborhood of the point a, from Theorem 4.2, $q = q_1\omega$, $r = r_1\eta$, where q_1 and r_1 are distinguished pseudopolynomials of the ring $\mathfrak{O}_a^{(n-1)}[z_n - a_n]$, and ω and η are divisors of unity of the ring $\mathfrak{O}_a^{(n)}$. Therefore we find that for the values of z in question $p = q_1 r_1 \omega\eta$; here $q_1 r_1$ is a distinguished pseudopolynomial of the ring $\mathfrak{O}_a^{(n-1)}[z_n - a_n]$, and $\omega\eta$ is a divisor of unity of the ring $\mathfrak{O}_a^{(n)}$. Then $\omega\eta = 1$, $p = q_1 r_1$, since from Theorem 4.2

the representation of the function $p \in \mathfrak{D}_a^{(n)}$ in the form of the product of a distinguished pseudopolynomial and a divisor of unity is unique.

THEOREM 4.5. *Each function $f \in \mathfrak{D}_a^{(n)}$ (in some sufficiently small neighborhood of the point a) decomposes in a unique way into the product of irreducible functions belonging to the ring $\mathfrak{D}_a^{(n)}$ (uniqueness up to equivalent multipliers).*

For functions $f \in \mathfrak{D}_a^{(n)}$ regular at the point $z = a$ relative to one of the variables, this theorem follows from the fact that a distinguished pseudopolynomial decomposes in a unique way into the product of irreducible distinguished pseudopolynomials with centers at the same point as the original one.

Indeed, if the product of two distinguished pseudopolynomials qr is divided by a distinguished irreducible pseudopolynomial g, then at least one of the factors must be divided by g (the centers of these pseudopolynomials coincide). This last is established by application to our pseudopolynomial of the algorithm of finding the greatest common divisor. [1]

In case the function $f \in \mathfrak{D}_a^{(n)}$ is not a regular function at the point a relative to any of the variables, we apply the transformation indicated in Theorem 4.3. Then the assertion of Theorem 4.5 for this function will follow from the invariance of the expansion into factors for such transformations.

THEOREM 4.6. *If in some neighborhood U_a of the point $z = a$ the zeros of the irreducible pseudopolynomial $q \in \mathfrak{D}_a^{(n-1)}[z_n - a_n]$ are zeros of the distinguished pseudopolynomial $p \in \mathfrak{D}_a^{(n-1)}[z_n - a_n]$, then the pseudopolynomial q is a factor in the expansion of the pseudopolynomial p into irreducible factors.*

PROOF. Suppose that the pseudopolynomial q does not appear as a factor of the pseudopolynomial p. Then these pseudopolynomials more generally have no common multipliers, since the pseudopolynomial q is irreducible. Using the process of finding the greatest common divisor, we determine pseudopolynomials $\lambda, \mu \in \mathfrak{D}_a^{(n-1)}[z_k - a_k]$, for which $\lambda p + \mu q = r$, while $r \in \mathfrak{D}_a^{(n-1)}$, $r \not\equiv 0$. This last, however, contradicts the hypotheses of our theorem.

For each point $(z_1', \cdots, z_{n-1}')$ of some sufficiently small neighborhood of

1) For analogous considerations for polynomials see for example the book by M. Bôcher, *Introduction to higher algebra*, The MacMillan Co., New York, 1929; p. 174ff.

the point $(a_1, \cdots, a_{n-1})$ in the space of the variables $z_1, \cdots, z_{n-1}$ one may find a value z_n' of the variable z_n such that the point $(z_1', \cdots, z_n') \in U_a$, $p(z_1', \cdots, z_n') = 0$, and accordingly also $q(z_1', \cdots, z_n') = 0$, $r(z_1', \cdots, z_{n-1}')=0$. The contradiction thus obtained forces us to reject the above supposition.

From the theorem just proved we readily obtain the following corollary, which will be used frequently.

COROLLARY. *If the functions* $f, g \in \mathfrak{O}_a^{(n)}$, *where* $f(a) = g(a) = 0$, *and if also in some neighborhood of the point* a *the following two conditions are satisfied:* 1) *the zeros of these functions coincide;* 2) $f = qf_1$, *where* $q, f_1 \in \mathfrak{O}_a^{(n)}$, $q(a) = 0$ *and the function* q *is irreducible, then in some neighborhood of the point* a *we also have* $g = qg_1$, *where the function* $g_1 \in \mathfrak{O}_a^{(n)}$.

The important difference between the case of $n + 1$ variables $(n > 1)$ and the case of two variables appears on consideration of the discriminant set of a pseudopolynomial. As is known from algebra, a polynomial has multiple roots if its discriminant is equal to zero. The discriminant of a polynomial, in our case of a pseudopolynomial, is the resultant of the polynomial itself and its derivative. In the case of the pseudopolynomial (1.54) we must consider the resultant of the pseudopolynomials:

$$f = w^m + A_1 w^{m-1} + \cdots + A_m,$$

$$f_w' = mw^{m-1} + A_1 (m-1) w^{m-2} + \cdots + A_{m-1}.$$

This resultant is formed from the coefficients of the pseudopolynomials by means of addition, subtraction and multiplication and therefore, like the coefficients themselves, is a holomorphic function $D(z_1, \cdots, z_n)$.

If f is a distinguished pseudopolynomial with center at the origin of coordinates and with $m > 1$, the value $w = 0$ is a multiple root of f for $z_1 = \cdots \cdots = z_n = 0$, and therefore the discriminant $D(0, \cdots, 0) = 0$. For $n = 1$ the discriminant $D(z)$ has only isolated roots. In this case, in a neighborhood of the origin of coordinates, the discriminant $D(z) \neq 0$ and the function f has only simple roots there. But this statement can not be made in the general case. For example, for $n = 2$ the discriminant is a holomorphic function of two variables. From the preparation theorem of Weierstrass it follows that now at any distance from the origin there are points at which the discriminant vanishes.

One may however show that, excluding from a sufficiently small

neighborhood U of the center of a distinguished pseudopolynomial the points at which its discriminant vanishes (we will call the collection of such points the *discriminant set* of the pseudopolynomial), we obtain a region. This follows from general topological facts. One can reach the same conclusion on the basis of the following theorem, which we shall prove for the case $n = 2$.

THEOREM 4.7. *Suppose that the coordinates of the point $P(w_0, z_0)$ satisfy the equation $q(w, z) = 0$, where $q(w, z)$ is a holomorphic function of its variables. If from a neighborhood of the point P we exclude all the points satisfying the equation $q(w, z) = 0$, then the remaining points of this neighborhood form a region.*

PROOF. We suppose that the point P is the origin of coordinates. Then, from the preparation theorem of Weierstrass, in a neighborhood of the point P

$$q(w, z) = z^{\mu}(w^m + a_1(z)\, w^{m-1} + \cdots + a_m(z))\,\Omega(w, z); \tag{1.57}$$

here $a_k(z)$, $\Omega(w, z)$ are holomorphic functions of their variables for $|z| < h$, $|w| < g$; for these values w, z the quantity $\Omega(w, z)$ does not vanish. The number h will be chosen so that for $|z| < h$ all the roots $w_k(z)$ of our pseudopolynomial will satisfy the condition $|w_k(z)| < g'$, where $0 < g' < g$. Suppose further that ζ is any point of the disk $|z| < h$. For each such ζ we exclude from the bicylinder $T\{|w| < g,\ |z| < h\}$ all closed rectilinear segments $\overline{l_k}(\zeta)$ defined by the conditions $z = \zeta$, $w = w_k(\zeta) + it$ $(0 \le t \le t_0)$, where t_0 is chosen so that $|w(\zeta) + it_0| = g'$. Here $k = 1, \cdots, m$. Moreover, if $\mu > 0$, we exclude from the bicylinder T the portion of the plane $z = 0$ for which $|w| < g$. We denote the set of all excluded points by Σ. Evidently the set Σ contains all the points of the bicylinder T whose coordinates satisfy the condition $q(w, z) = 0$. We denote by T^* the set of points remaining in the bicylinder T after the removal of the set Σ.

We shall prove first that T^* is a region. Indeed, any two points (w', z') and (w'', z'') of the set T^* may be joined by a line lying in T^*. Evidently $z' \neq 0$, $z'' \neq 0$ (if $\mu = 0$, there is no need to exclude zero values of z). If $|w'|$, $|w''| > g'$, then our assertion is evident, since all points of the bicylindrical region $T_1\{0 < |z| < h,\ g' < |w| < g\}$ belong to T^*. If one or both numbers $|w'|$, $|w''|$ are less than g', then from the method of construction of Σ it follows that along with the point (w', z') (or (w'', z'')) there remains also the segment parallel to the coordinate axis $z = \mathrm{Re}(w) = 0$, whose direction is

opposite to $l_k(\zeta)$ and which joins that point to T_1. Thus T^* is a connected set. It consists of interior points and therefore forms a region.

We must now join to it the points of the segments $\overline{l}(\zeta)$ not satisfying the equation $q(w, z) = 0$. In the disk $|w| < g'$, on the plane $z = \zeta$, we have drawn a finite number m of such segments. If M is some point of the segment $\overline{l}_k(\zeta)$, distinct from the point $(w_k(\zeta), \zeta)$, then, moving from it along a perpendicular to $\overline{l}_k(\zeta)$ in the plane $z = \zeta$, avoiding the points $(w_k(\zeta), \zeta)$, which may occur on this path, we join the point M to the set T^*. Also, since the set of points $E\{q(w, z) = 0\}$ is closed, we may consider it as established that the points of the bicylinder T for which $q(w, z) \neq 0$ form a region. Our theorem is proved.

3. **Bundles of rings of germs of holomorphic functions.** The functions $f(z)$ holomorphic in some region $D \subset C^n$ form a ring of integrity $\mathfrak{O}_D$. On the other hand, the functions holomorphic in some open set $D \subset C^n_z$ consisting of several components not connected to one another form a commutative ring $\mathfrak{O}_D$ which is not a ring of integrity. For example, if the set D consists of two regions D' and D'' not connected to each other, then by putting

$$f_1(z) = \begin{cases} 0 \text{ for } z \in D', \\ 1 \text{ for } z \in D'', \end{cases} \quad f_2(z) = \begin{cases} 1 \text{ for } z \in D', \\ 0 \text{ for } z \in D'', \end{cases}$$

we obtain two different functions distinct from the identically zero function, with $f_1, f_2 \in \mathfrak{O}_D$, whose product is zero. These functions are divisors of zero and in the case under discussion the ring $\mathfrak{O}_D$ is not a domain of integrity.

In order to characterize the set of functions holomorphic at the points of some region, or in the general case, of an open set D, we make use of the concept of bundle (see Introduction, p. 14). We consider the bundle of rings of germs of holomorphic functions $\mathfrak{O}(D) = \{\mathfrak{O}_z, z \in D\}$ over the set $D \subset C^n_z$. A germ f_z of a holomorphic function is defined in the same way as the germ of a continuous function. But here the situation is simpler than with continuous functions, since a holomorphic function coinciding with the holomorphic function $f(z)$ in some neighborhood of the point z_0 coincides with it in the whole region of convergence of its power series about the point z_0. Therefore the concept of the germ of a holomorphic function is not different from the concept of a holomorphic functional element, as we said in subsection 2 of §1. In a corresponding way also the concept of a neighborhood in the space of the bundle of rings of holomorphic functions $\mathfrak{O}(D) = \{\mathfrak{O}_z, z \in D\}$ is simpler.

4. Analytic sets. [1]

DEFINITION. Let B be an open set in the space C_z^n of complex variables $z_1, \cdots, z_n$. A subset m of the set B will be said to be *analytic at the point* $z \in B$ if there exists a neighborhood $U_z \subset B$ such that the set $m \cap U_z$ coincides with the set of common zero points of some finite set of functions holomorphic in that neighborhood U_z. The set m is said to be *analytic in the open set* B if it is analytic at all of its points.

If the set m is analytic at the point $z \in B$, then the collection of holomorphic functional germs (elements), vanishing at the points of the set $m \cap U_z$, is an ideal I_{U_z} in the ring of integrity $\mathfrak{D}_{U_z}$ of all functional germs holomorphic in the neighborhood U_z of the point z. Indeed, if on the set $m \cap U_z$ one has the equations $f_1 = \cdots = f_p = 0$, then the equation $\lambda_1 f_1 + \cdots + \lambda_p f_p = 0$ also holds there, with $f, \lambda \in \mathfrak{D}_{U_z}$. The ideal consisting of functions of the form $\lambda_1 f_1 + \cdots + \lambda_p f_p$ is denoted by the symbol $[f_1, \cdots, f_p]$. The functions $f_1, \cdots \cdots, f_p$ are called a *basis* of that ideal if they are linearly independent.

Conversely, the common zeros of holomorphic functions belonging to some ideal $I_{U_z} \subset \mathfrak{D}_{U_z}$ always form in the neighborhood U_z an analytic set. We will say that it is defined by the ideal I_{U_z}.

Considering different neighborhoods U_z of the point z we obtain, in the ring $\mathfrak{D}_z$ of functional germs that are holomorphic at the point z, an ideal I_z consisting of *all* germs each of which vanishes at evéry point of the set m within the corresponding neighborhood of the point z. The ideal I_z is called the *proper* ideal of the set m at the point z or more concisely the ideal of the set m at the point z. It remains to be noted that the set m can be defined in the neighborhood of the point z by an ideal other than its proper one. For example, for the set given by the ideal $[z_1^2]$ the proper ideal is $[z_1]$.

Above we have defined an analytic set at the point $z \in B$ by means of a *finite* set of functions which vanish on the set and are holomorphic in some neighborhood of the point z. Now we shall show that the requirement of finiteness of this set is unnecessary.

THEOREM 4.8. *Every ideal I_ζ in the ring $\mathfrak{D}_\zeta$ of functions holomorphic at some point $\zeta \in C^n$ has a finite basis.*

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1) In subsections 4 and 5 we present the simplest properties of analytic sets. More information on analytic sets will be found in §§14–16 of Chapter III.

REMARK. A ring in which every ideal has a finite basis is said to be a *Noetherian ring.* Theorem 4.8 asserts that the ring $\mathfrak{O}_\zeta$ of functions holomorphic at some point $\zeta \in C^n$ is Noetherian.

PROOF.[1] We let $\mathfrak{O}_\zeta^{(n)}$, or for brevity $\mathfrak{O}^{(n)}$, denote the ring of functions holomorphic at the point $\zeta \in C^n$, and let $I_\zeta^{(n)}$, or for brevity $I^{(n)}$, denote an ideal in that ring. Evidently, if $n = 0$ the ring $\mathfrak{O}^{(n)}$ reduces to the ring of constants $\mathfrak{O}^{(0)}$, which is obviously Noetherian. We shall show that if the rings $\mathfrak{O}_\zeta^{(k)}$ $(k = 1, \cdots, n-1)$ are also Noetherian, then the ring $\mathfrak{O}_\zeta^{(n)}$ is too.

We move the point ζ to the origin of coordinates. If the ideal $I^{(n)}$ contains some nonzero element, then we may suppose that it contains also a function $f_0(z_1, \cdots, z_n)$ such that $f_0(0, \cdots, 0, z_n) \not\equiv 0$. If this is not so, we apply to the variable z an appropriate transformation as in Theorem 4.3. As a result, we obtain an ideal containing at least one function with the indicated property. Evidently the desired ideal will have a finite basis if the transformed ideal has one. The notation $I^{(n)}$ will from here on indicate this latter case only. We consider a distinguished pseudopolynomial that is equivalent in the neighborhood of the point ζ to the function we have chosen, and we shall denote it by f_0. Evidently the pseudopolynomial $f_0 \in I^{(n)}$. Suppose that z_n^r is its leading term, and that $\chi_s(z_1, \cdots, z_{n-1})$ $(s = 1, \cdots)$ are its roots, considered in some neighborhood of the origin of coordinates.

Any element of the ring $\mathfrak{O}^{(n)}$, in particular any function $f \in I^{(n)}$, may be represented in the form $f = f_0 F_0 + f_{r-1}^*$, where $F_0 \in \mathfrak{O}^{(n)}$ and

$$f_{r-1}^* = f_{r-1}^*(z_n) = b_{r-1} z_n^{r-1} + \cdots + b_0,$$

with the coefficients $b_k \in \mathfrak{O}^{(n-1)}$, $k = 0, \cdots, r-1$. Here the pseudopolynomial $f_{r-1}^*(z_n)$ is defined, and indeed uniquely, by the conditions

$$f_{r-1}^*(\chi_s) = f(z_1, \cdots, z_{n-1}, \chi_s), \quad s = 1, 2, \cdots$$

(for multiple roots χ_s the missing conditions are replaced by the requirement that the corresponding derivatives be equal).

Indeed, choose the function $F_0 = (f - f_{r-1}^*)/f_0$. Under our conditions it is evidently possible to extend it to the points $z_n = \chi_s(z_1, \cdots, z_{n-1})$ in such a

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1) The idea of this proof is due to Hilbert. See, for example, W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry,* Vol. 1, Macmillan, New York, 1947.

way that it turns out to be holomorphic in an entire neighborhood of the origin of coordinates. Therefore our assertion follows.

We consider the set A_{r-1} of pseudopolynomials f_{r-1}^*, constructed for the various functions $f \in I^{(n)}$. It is easily verified that the set of their coefficients $\{b_{r-1}\}$ is an ideal $I^{(n-1)}$ in the ring $\mathfrak{O}^{(n-1)}$. In view of our hypotheses this ideal $I^{(n-1)}$ has a basis $\phi_1, \cdots, \phi_{r_0}$. Denote by f_s the pseudopolynomial from the set A_{r-1} with the highest coefficient ϕ_s $(s = 1, \cdots, r_0)$. Then we have the representation

$$f = f_0 F_0 + \sum_{s=1}^{r_0} F_s f_s + f_{r-2}^*,$$

where f_{r-2}^* is a pseudopolynomial of degree not higher than $r-2$ in the variable z_n.

Continuing in this way, after a finite number of steps we shall indeed have constructed a finite basis in the ideal $I^{(n)}$.

We return to the consideration of analytic sets. From our definition it results that a set m analytic in an open set B is always closed relative to that set.

We define an open subset of the set m as the intersection $m \cap U$, where U is an open set of the space C_z^n. As a result of this induced topology the set m becomes a topological space. One may show that it is always locally compact, locally linearly connected and has a countable basis of open sets.

As we have seen, if the set m is analytic at a point $z \in B$, then the collection of holomorphic functional germs, each of which vanishes at the points of some set $m \cap U_z$, forms an ideal I_z in the ring of integrity $\mathfrak{O}_z$. If the set m is analytic in the entire open set B, then the union $\{I_z, z \in B\}$ is a bundle of ideals constituting in their turn a subbundle of the bundle of rings of germs of holomorphic functions: $\mathfrak{O}(B) = \{\mathfrak{O}_z, z \in B\}$.

An analytic set m defined in an open set $B \subset C_z^n$ has at each point $z \in m$ some topological dimension $\mathrm{Dim}_z(m)$.[1] This topological dimension is always an even number. We shall call *the complex dimension of the set* m *at the point* z the quantity $d_z(m) = \frac{1}{2}\mathrm{Dim}_z(m)$. Finally, we shall call the quantity $d(m) = \max_{z \in m} d_z(m)$ the *maximal complex dimension* or simply the complex dimension

1) See on this topic §2 of the Introduction.

of the set m in the open set B. One should note that always $d(m) \leq n$.

In many cases it is convenient also to consider *the complex codimension of the analytic set* m at the point z, namely, the quantity $c_z(m) = n - d_z(m)$, and the (minimal) complex codimension of the analytic set m in the open set B, which is the quantity $c(m) = \min_{z \in m} c_z(m)$.

The analytic set m is said to be *homogeneous* or *purely-dimensional* in the open set B if at all points $z \in B$ we have the equation $d_z(m) = d(m)$.

One may show that if m_1 and m_2 are analytic sets in the open set B, while $m_1 \subset m_2$ and $d_z(m_1) < d_z(m_2)$ for all points $z \in m_1$, then the set m_1 is nowhere dense in the set m_2. In particular, if $d_z(m_2) = n$ and $d_z(m_1) < n$, then the set m_1 is nowhere dense in the open set B.

The so-called sparse sets, which play an important role in the theory of functions, have the indicated property.

DEFINITION (sparse, almost sparse set). A set $N \subset B$ is said to be *sparse* in the open set B if it is closed in B and if each point $z \in B$ has a neighborhood U_z such that the set $N \cap U_z$ is contained in some analytic set $M_z \subset U_z$ which is nowhere dense in U_z. The set $N \subset B$ is said to be almost sparse in the open set B if it is the union of a countable set of sparse sets.

We observe that whether or not a set is analytic depends on the choice of cartesian coordinates used in the space $R_{2n} = C^n$. However, it clearly remains invariant if a transformation of the system of coordinates is realized by means of holomorphic functions:

$$z_k = \sum_{l=1}^{n} a_{kl} Z_l + b_k, \quad k = 1, \cdots, n.$$

Here the quantities a_{kl} are subjected to conditions expressing the independence of the distance between two arbitrary points from the choice of old and new systems of coordinates. [1)]

5. Reducible and irreducible analytic sets. Analytic surfaces. An analytic set m is said to be *reducible* in the open set B if it can be represented as

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1) In order to obtain these conditions one may put $b_k = 0$ and express the fact that the distances from the origin of coordinates to the points z and Z are the same. For example, for $n = 2$ we observe that for any z_1, z_2 we must have $|z_1|^2 + |z_2|^2 = |a_{11} Z_1 + a_{12} Z_2|^2 + |a_{21} Z_1 + a_{22} Z_2|^2$. By comparing the coefficients of similar terms we obtain these conditions explicitly.

$m_1 \cup m_2$, where m_k $(k = 1, 2)$ are nonempty analytic sets in B different from the set m. In the contrary case the set m is said to be *irreducible* in the set B.

Suppose that m is a set which is analytic at the point $z \in C^n_z$. The collection of all sets which are analytic at the point z and coincide with the given set m in respective neighborhoods (generally different for different sets) of the point z is called *the germ of the analytic set at the point* z and is denoted by m_z. We will say that each of these sets (the original set m included) *represents* the germ m_z at the point z, and the germ m_z *belongs* to each of these sets. The germ m_z is said to be *empty* if the intersection $m \cap U_z$, where U_z is some neighborhood of the point z, is empty.

We shall further consider the collection $M_z = \{m_z\}$ of all germs of analytic sets at the point z and in addition the union M of these collections for all points $z \in B$.

We introduce a topology into the union M as follows. Suppose that U_{z_0} is a neighborhood of the point $z_0 \in B$, and that $f_1, \cdots, f_r$ are holomorphic functions in the neighborhood U_{z_0}, where $f_1(z_0) = \cdots = f_r(z_0) = 0$. Let the set $\{f_1 = 0, \cdots, f_r = 0\}$ represent the germ m_{z_0} at the point z_0. By the neighborhood $V_{m_{z_0}}$ we shall understand the collection of germs $m_{z'}$ represented at the points $z' \in U_{z_0}$ by the sets $\{f_1(z) - f_1(z') = 0, \cdots, f_r(z) - f_r(z') = 0\}$. These sets $V_{m_{z_0}}$ form a basis for the topology in the set M. It is easy to verify that Axiom I for the definition of a bundle is satisfied for this topology. Thus we obtain a bundle $M(B) = \{M_z, z \in B\}$, consisting of the collection M_z of germs m_z over the open set B.

A germ of the analytic set p_z is said to be prime if it cannot be separated into two nonempty germs of analytic sets distinct from itself at the point z.

We say that the germ p_z is separated into the germs $p_z^{(1)}$ and $p_z^{(2)}$ if for each three sets m, $m^{(1)}$, $m^{(2)}$ representing these germs in the neighborhood V_z of the point z, one may indicate a neighborhood $U_z \subset V_z$ such that $m \cap U_z = (m^{(1)} \cap U_z) \cup (m^{(2)} \cap U_z)$.

A set m analytic in an open set B is said to be *irreducible* at the point $z \in m$ if it represents at that point a prime germ of an analytic set. This happens if and only if there exists a basis of neighborhoods $\{U_z^{(\nu)}, \nu = 1, 2, \cdots\}$ of the point z such that for each ν the intersection $m \cap U_z^{(\nu)}$ is an analytic set irreducible in that neighborhood.

The set m is said to be *locally irreducible* at the point $z \in m$ if it is irreducible at all points $\zeta \in m \cap U_z$ where U_z is some neighborhood of the point z. A prime germ of the analytic set p_z is said to be *locally irreducible* if it is at the point z a locally irreducible analytic set m.

We present some simple examples in the space C^3 of variables w, z_1, z_2. 1) The analytic set $m_1\{w^4 - z_1^2 z_2^2 = 0\}$ is reducible at the origin of coordinates, since it may be represented in the neighborhood of the origin as the union of sets $m^{(1)}\{w^2 - z_1 z_2 = 0\}$, $m^{(2)}\{w^2 + z_1 z_2 = 0\}$. 2) The analytic set $m_2\{w^2 - z_1^2 z_2 = 0\}$ is irreducible at the origin of coordinates; however it is not locally irreducible there, since in the neighborhood of any of its points P with coordinates $w = z_1 = 0$, $z_2 \neq 0$ it may be represented in the form of the union of sets $m_2^{(1)}\{w - z_1 \times (\sqrt{z_2})_1 = 0\}$, $m_2^{(2)}\{w + z_1(\sqrt{z_2})_1 = 0\}$. Here $(\sqrt{z_2})_1$ is one of the holomorphic branches of $\sqrt{z_2}$ in the neighborhood of the point P. 3) The analytic sets $m_3\{w = 0\}$, $m_4\{w^3 - z_1^2 = 0\}$, $m_5\{w^2 - z_1 z_2 = 0\}$ are locally irreducible at the origin of coordinates.

THEOREM 4.9. *In order that the analytic set m should be irreducible at the point z, it is necessary and sufficient that its proper ideal I_z should be prime.* [1]

PROOF. 1) Suppose that the proper ideal I_z of the irreducible set m at the point z is not prime: let $\phi_1, \phi_2 \in \mathfrak{O}_z$; $\phi_1, \phi_2 \notin I_z$, but $\phi_1 \phi_2 \in I_z$. We shall show that then in some neighborhood U of the point z we must have $m \cap U = m^{(1)} \cup m^{(2)}$, which contradicts the hypothesis that the set m is irreducible. Here $m^{(1)}$ and $m^{(2)}$ are analytic sets defined in the neighborhood U by the ideals $I_z + [\phi_1]$, $I_z + [\phi_2]$. [2] In view of our suppositions $m^{(1)} \neq m \cap U$, $m^{(2)} \neq m \cap U$.

We form the intersection of the ideals $I_z + [\phi_1]$ and $I_z + [\phi_2]$. Suppose that $f_1 + \alpha_1 \phi_1 = f_2 + \alpha_2 \phi_2$ (where $f_1, f_2 \in I_z$; $\alpha_1, \alpha_2 \in \mathfrak{O}_z$) is an element of that intersection. Then $\alpha_1 \phi_1 = f + \alpha_2 \phi_2$, where $f = f_2 - f_1 \in I_z$, $\alpha_1^2 \phi_1^2 = \alpha_1 \phi_1 f + \alpha_1 \alpha_2 \phi_1 \phi_2 \in I_z$, since $f \in I_z$ and $\phi_1 \phi_2 \in I_z$. On the other hand, I_z is a proper ideal of the set m. Therefore along with the element $\alpha_1^2 \phi_1^2$ there

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1) The ideal I_z is said to be prime if for $fg \in I_z$ and $f \notin I_z$ one always has $g \in I_z$

2) By $I_z + [\phi]$ (where $\phi \in \mathfrak{O}_z$) we denote an ideal consisting of all elements of the form $f + \lambda\phi$. Here $f \in I_z$ and λ is any element of the ring $\mathfrak{O}_z$.

also belongs to it the element $\alpha_1 \phi_1$, and along with them also the element $f_1 + \alpha_1 \phi_1 = f_2 + \alpha_2 \phi_2$.

Therefore it results that the ideal $I_z = (I_z + [\phi_1]) \cap (I_z + [\phi_2])$, and the set $m \cap U = m^{(1)} \cup m^{(2)}$, which, as we have already said, contradicts the hypotheses of the theorem.

2) Suppose that the analytic set m with a prime proper ideal I_z is reducible at the point z. Suppose that $m \cap U = m^{(1)} \cup m^{(2)}$, where U is some neighborhood of the point z, and $m^{(1)} \neq m \cap U$, $m^{(2)} \neq m \cap U$ are analytic sets in that neighborhood. We denote by $I_z^{(1)}$ and $I_z^{(2)}$ the proper ideals of these sets, so that $I_z = I_z^{(1)} \cap I_z^{(2)}$. Let $f_1 \in I_z^{(1)}$ but $f_1 \notin I_z^{(2)}$, and $f_2 \in I_z^{(2)}$ but $f_2 \notin I_z^{(1)}$. Then the product $f_1 f_2$ belongs to both of these ideals, and accordingly also to the ideal I_z, although neither f_1 nor f_2 belongs to it. This conclusion contradicts the hypothesis that the ideal I_z is prime.

It is known from algebra that [1)] any ideal of a Noether ring (i.e., a ring for which Theorem 4.8 on bases for ideals holds) may be represented as the intersection of a finite set of primary ideals [2)] $\{I\}$.

The intersection of two primary ideals to which there corresponds one and the same prime ideal again turns out to be a primary ideal, to which the same prime ideal corresponds. Thus it is possible to choose the primary ideals $\{I\}$ in such a way that none of them can be omitted and that all the prime ideals corresponding to them are distinct. Evidently the analytic sets defined by primary ideals coincide with analytic sets defined by the prime ideals corresponding to them. In view of Theorem 4.9 an analytic set with a prime proper ideal is irreducible. One may show that a prime ideal is always proper for the analytic set defined by it. [3)] Thus, we arrive at the following theorem, which for future applications we formulate in terms of germs of analytic sets.

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1) See for example van der Waerden, *Modern algebra*, Part II, G. E. Stechert and Co., New York, 1943, §87, p. 32.

2) The ideal I is said to be primary if for $fg \in I$ and $f \notin I$ there always exists an integer k such that $g^k \in I$. A prime ideal is always primary. To each primary ideal there corresponds a prime ideal $\bar{I}$, consisting of those elements f for which $f^k \in I$. Here k is some integer defined for each element f. See on this subject van der Waerden, *Modern algebra*, Part II, p. 34ff.

3) The basis for this assertion may be found for example in Bochner-Martin [1], p. 240 ff.

THEOREM 4.10. *Every germ of an analytic set* m_z *may be uniquely represented (up to order) as the union of a certain number of prime germs of analytic sets*

$$m_z = \mathfrak{p}_z^{(1)} \cup \mathfrak{p}_z^{(2)} \cup \cdots \cup \mathfrak{p}_z^{(r)}.$$

These prime germs $\mathfrak{p}_z^{(k)}$ $(k = 1, \cdots, r)$ will be said to *belong to the germ* m_z and to the analytic sets which represent the germ m_z at the point z.

DEFINITION (ordinary point of an analytic set). The point z of some set m, analytic in the open set B, is said to be an *ordinary point of* m if 1) there exists a neighborhood $U_z \subset B$ such that the set $m \cap U_z$ coincides with the set of common zeros of some number r of functions holomorphic at that point z, and 2) the rank of the Jacobian of these functions at the point z is equal to $n - r$.

Evidently in this case $d_z(m) = n - r$. If z is an ordinary point of the analytic set m, then that set is locally irreducible at the point z.

The points of an analytic set which are not ordinary points are called its exceptional points. They must not be confused with the singular points of an analytic set (see their definition in subsection 4 of §6).

Let m be some analytic set in the open set B. We may show that the collection l of exceptional points of the set m is itself an analytic set in the same open set B, while $d_z(l) < d_z(m)$ for all points $z \in l$.

Let m_1 be an irreducible component of the set m. Then the collection of ordinary points of the set m_1 turns out to be a connected set. The set m is irreducible in the open set B if and only if the set of ordinary points of the set m is connected.

The set m is locally irreducible in the open set B if and only if the set l of exceptional points of the set m does not split the space m. Similarly the set m is irreducible at the point $z \in m$ if and only if the set l of exceptional points of the set m does not split the set m at the point z.

If the set m is irreducible at the point $z \in m$, then any analytic set $m' \subset m$ for which $d_z(m') < d_z(m)$ does not split the space of the set m at the point z.

A special case of analytic set is the analytic surface.

DEFINITION (analytic surface). A complex r-dimensional element of the surface Γ^r is said to be analytic if for each of its points z there is a neighborhood U_z such that $\Gamma^r \cup U_z$ is an analytic set irreducible at the point z.

Thus the analytic sets considered above, m_3, m_4, m_5, are two-complex-dimensional analytic sets in the neighborhood of the origin of coordinates.

In the particular case when the r-complex-dimensional element of a surface may be defined by $n - r$ independent linear equations $\Sigma_{s=1}^{n} a_{ks} z_s = 0$, $k = 1, \cdots, n - r$, it is a piece of an r-complex-dimensional analytic surface. If all the points of the space C^n whose coordinates satisfy the indicated conditions belong to this piece, it is said to be an r-complex-dimensional *analytic plane*.

We note that in many cases, instead of the complex dimension of an analytic surface, one gives its complex codimension.

At the ordinary point $P(z_1^0, \cdots, z_n^0)$ of the r-complex-dimensional analytic surface $\{ f_1 = 0, \cdots, f_{n-r} = 0 \}$ there is a tangent r-complex-dimensional analytic plane. It is defined by the equations

$$\sum_{k=1}^{n} \left(\frac{\partial f_\alpha}{\partial z_k}\right)_P (z_k - z_k^0) = 0, \quad \alpha = 1, \cdots, n - r. \tag{1.58}$$

Here $f_1, \cdots, f_{n-r}$ are holomorphic functions at the point z satisfying the requirements given in the definition of an ordinary point of the surface.

Observe that for the analytic surfaces m_4, m_5 the origin of coordinates is an exceptional point, and for the surface m_3 it is an ordinary point.

6. Some properties of analytic surfaces and sets. First of all we prove for the case of two variables two theorems which will be needed later.

THEOREM 4.11. *Two distinct surfaces which are analytic in all of their points that belong to some closed bounded region D can intersect there only in a finite set of points.*

PROOF. In the contrary case there would have to be a point of intersection of the surfaces (w', z') which is a limit point of a sequence of intersection points (w_n, z_n). In some neighborhood of (w', z'), in view of the preparation theorem of Weierstrass, the equations of our analytic surfaces may be replaced by equations $h(w, z) = 0$, $g(w, z) = 0$, where $h(w, z)$ and $g(w, z)$ are irreducible distinguished pseudopolynomials with center at the point z'. Since the surfaces are distinct, we can use the algorithm for finding the greatest common divisor to find pseudopolynomials $p(w, z)$, $q(w, z)$ and a function $r(z)$ such that in that neighborhood

$$p(w, z) h(w, z) + q(w, z) g(w, z) = r(z). \tag{1.59}$$

Therefore it follows that $r(z_n) = 0$, and therefore, from the uniqueness theorem for functions of one variable, more generally $r(z) \equiv 0$. This contradicts our hypothesis on the irreducibility and difference of the pseudopolynomials $h(w, z)$ and $g(w, z)$. Our theorem is proved.

From this theorem it follows that in the space C^2 two analytic surfaces coincide if they have in common some sequence of points along with their limit point.

THEOREM 4.12. *In the neighborhood of any point $P(w_0, z_0)$ belonging to the analytic surface $f(w, z) = 0$, this surface may be given by the equation*

$$w = w_0 + \sum_{k=1}^{\infty} a_k (z - z_0)^{\frac{k}{m}}.$$

Here m is some integer.

PROOF. Because of the preparation theorem of Weierstrass, in the neighborhood of the point P the function $f(w, z)$ may be represented by formula (1.48) (we take $\mu = 0$, since the function $f(w, z)$ is supposed irreducible):

$$f(w, z) = [(w - w_0)^m + A_1(z)\ (w - w_0)^{m-1} + \cdots + A_m(z)]\,\Omega(w, z). \quad (1.60)$$

Therefore it follows that the equation $f(w, z) = 0$ in the neighborhood of the point P is equivalent to the equations

$$w = w_k(z); \quad k = 1, \cdots, m.$$

The functions $w_k(z)$ are holomorphic in some neighborhood of the point z_0, aside from the point $z = z_0$ itself (by the preceding theorem there are no other branch points of these functions in the indicated neighborhood).

The roots $w_1(z), \cdots, w_m(z)$ may be considered as the values of one (univalent) holomorphic function of a new variable ζ. To this end we have to put $z = \zeta^m + z_0$. On a full turn around the origin in the plane of ζ the point z goes m times around the point $z = z_0$. Considering the values of some root, for example $w_1(z)$, along this path, we obtain, as a result of each such turn in the z plane, the values of a new root at the desired point, and after m turns we obtain the initial root. In the contrary case, if the initial root is obtained after p turns ($p < m$), then this would mean that $w_1, \cdots, w_p$ form a closed system, so that their symmetric functions would be univalent. Then the pseudopolynomial (1.60) would be divided, from Theorem 4.3, by the pseudopolynomial

$(w - w_1) \cdots (w - w_p)$ and would not be irreducible. Therefore it follows that the root $w_1(z) = w_1(\zeta^m)$, considered as a function of ζ, is a univalent holomorphic function of ζ and takes on successively, on a full turn of ζ around the origin, the values of all the roots $w_k(z)$ on the corresponding closed path in the z plane. Therefore, in some neighborhood of the origin

$$w = w_0 + \alpha_1 \zeta + \alpha_2 \zeta^2 + \cdots.$$

Returning to the original variable z by the substitution $\zeta = (z - z_0)^{1/m}$, we obtain

$$w = w_0 + \sum_{k=1}^{\infty} \alpha_k (z - z_0)^{\frac{k}{m}}$$

With this our theorem is proved.

We note that if P is an ordinary point of the surface, then $m = 1$.

For an analytic set of general form there is a theorem which we present without proof.[1]

THEOREM 4.13. *Suppose that $B_k^{(1)}$ and $B_{n-k}^{(2)}$ are regions lying respectively in the spaces C^k of the variables $z_1, \cdots, z_k$ and C^{n-k} of the variables $z_{k+1}, \cdots, z_n$, and m is a pure k-complex-dimensional analytic set, lying in the product of the regions $B_k^{(1)} \times B_{n-k}^{(2)}$. Suppose that the closed region $B_{n-k}^{(2)}$ is compact in the space C^{n-k}, and that the closure $\overline{m}$ of the set m in the space $C^n = C^k \times C^{n-k}$ has no points in common with the set $B_k^{(1)} \times \partial B_{n-k}^{(2)}$ (where $\partial B_{n-k}^{(2)}$ is the boundary of the region $B_{n-k}^{(2)}$).*

Then there exist polynomials

$$\omega_q(z_q, z_1, \cdots, z_k) = z_q^{m_q} + \sum_{\mu=0}^{m_q - 1} A_q^{\mu}(z_1, \cdots, z_k)\, z_q^{\mu},$$

$$q = k+1, \cdots, n,$$

having no multiple roots, with coefficients $A_q^{(\mu)}(z_1, \cdots, z_k)$ holomorphic in the region $B_k^{(1)}$, such that the set m may be made up from irreducible components of the analytic (in the region $B_k^{(1)} \times B_{n-k}^{(2)}$) set

$$\{ z \in B_k^{(1)} \times B_{n-k}^{(2)},\ \omega_{k+1} = 0, \cdots, \omega_n = 0 \}.$$

1) See Remmert [1] and Remmert-Stein [1].

If the set m is irreducible in the region $B_k^{(1)} \times B_{n-k}^{(2)}$, then all the polynomials ω_q may also be taken to be irreducible.

In the case when the set m is pure-dimensional and k its complex dimension, one may select for each point $z \in m$, after an appropriate linear mapping, a polycylindrical neighborhood $\mathfrak{S}_k^{(1)} \times \mathfrak{S}_{n-k}^{(2)}$ (where $\mathfrak{S}_k^{(1)} = \{|z_q'| < r_q,$ $q = 1, \cdots, k\}$; $\mathfrak{S}_{n-k}^{(2)} = \{|z_q'| < r_q,\ q = k+1, \cdots, n\}$, $z_1', \cdots, z_n'$ are the coordinates of the points of the space after the linear transformation; for these coordinates the point z becomes the origin) such that all the hypotheses of Theorem 4.13 are satisfied for the analytic set $m \cap (\mathfrak{S}_k^{(1)} \times \mathfrak{S}_{n-k}^{(2)})$.

7. Traces of functions and forms on analytic sets. An important role further on is played by the traces (restrictions) $f|_m$ of holomorphic functions $f(z)$, where $z \in C^n$ on the analytic sets $m \subset C^n$. Essential use is made of them in the theory of complex spaces (see §§16–17 of Chapter III).

At present we confine ourselves to the observation that for such traces $f|_m$ there is a maximum principle. If $\zeta \in m$ is an ordinary point of the analytic set $m \subset C^n$ and the function $f(z)$ is holomorphic in some neighborhood $U_\zeta \subset C^n$, then the modulus of the function $f|_m$ is either constant on $U_\zeta \cap m$ or else cannot take on its maximum value at the point ζ.

Indeed, in view of Theorem 2.3, the function $f|_m$ in the neighborhood of the point ζ reduces to a holomorphic function of q complex variables in the neighborhood of some point $\tilde{\zeta} \in C^q$, where $q = d_\zeta(m)$. Therefore, from Theorem 3.10, our assertion follows.

DEFINITION (sets for which the maximum principle holds). Consider a subspace m of the space C^n (m need not be an analytic set) with a topology induced by the topology of the space C^n. Suppose: 1) $b \subset m$ is any bounded region in the sense of the topology of the space m; 2) the function $f(z)$ is holomorphic in the neighborhood $U(b_0) \subset C^n$ of any region b_0 if $\overline{b}_0 \subset b$; 3) the modulus of the function $f|_m$ is continuous in the closed region $\overline{b}$. If under the indicated conditions the modulus of the function $f|_m$ has its largest value in the closed region $\overline{b}$ on its boundary, then one says that the maximum principle holds for the set m.

In what follows we shall also consider sets for which the maximum principle holds for some class of holomorphic functions.

From our propositions one may conclude that the maximum principle holds

for analytic sets consisting of ordinary points.

Now we consider a pure-dimensional analytic surface $m \subset C^n$, $d(m) = q$. First we suppose that this surface consists entirely of ordinary points. Then, in view of Theorem 2.3, on this surface in the neighborhood of each of its points some $n - q$ of the n differentials $dz_1, \cdots, dz_n$ may be expressed linearly in terms of the rest of them.

Now we choose an exterior differential form of degree k (with $k > q$) of the form (1.20_1):

$$\alpha = \sum_k^n{}' A_{i_1 \cdots i_k} dz_{i_1} \wedge \cdots \wedge dz_{i_k},$$

given in some region $B \supset m$, and we find its trace on the surface m. In each product $dz_{i_1} \wedge \cdots \wedge dz_{i_k}$ we consider on this surface $k - q$ factors linearly expressed in terms of the remaining q factors. In view of the rule of exterior multiplication of differentials such a product is equal to zero. Hence it follows that the entire trace of the form α on the surface m is equal to zero.

From what has been said it follows in particular that for integrals of the form (1.22_1)

$$\int_m \alpha = \int_m \sum_k^n{}' A_{i_1 \cdots i_k} dz_{i_1} \wedge \cdots \wedge dz_{i_k} = 0,$$

since in this case $k = 2q$ is the topological dimension of the surface m.

If the surface m_q has exceptional points, these constitute, as we have seen, an analytic set of complex dimension less than q. Therefore the result obtained for integrals of the type (1.22_1) easily extends to the case of analytic surfaces having exceptional points, by a limit process.

It holds also for analytic sets which are the union of some set of analytic surfaces.

8. Curvature of an analytic surface. Now we consider some differential-geometric properties of analytic surfaces. We begin by deriving some auxiliary formulas.

For each vector u of the space C^n of variables $z_k = x_k + iy_k$ $(k = 1, \cdots, n)$ we may, along with its real coordinates x^k, y^k $(k = 1, \cdots, n)$, consider also its complex coordinates $u^k = x^k + iy^k$, $\overline{u^k} = x^k - iy^k$ $(k = 1, \cdots, n)$. Each such vector defines an analytic plane U^1, in which there lies the bundle of vectors

$$w^k = \alpha u^k, \quad w^{\overline{k}} = \overline{\alpha} u^{\overline{k}}.$$

Here α is a complex number. Through each point $z^0(z_1^0, \cdots, z_n^0) \in C^n$ there passes only one analytic surface U^1 containing the vector u^α. The equations of this plane have the form

$$\frac{z_1 - z_1^0}{u^1} = \cdots = \frac{z_n - z_n^0}{u^n}.$$

For $n = 2$ the plane U^1 will be defined by the equation $z_2 - z_2^0 = \omega(z_1 - z_1^0)$, where $\omega = u^2/u^1$, if $u^1 \neq 0$ and by the equation $z_1 - z_1^0 = 0$ if $u^1 = 0$. The number ω will be called the *parameter* of the plane U^1 (for the plane $z_1 - z_1^0 = 0$ the parameter will have the value $\omega = \infty$). The plane U^1 itself will be called for brevity *the plane* ω. The angle ψ between the vectors u^k, v^k $(k = 1, \cdots \cdots, n, \overline{1}, \cdots, \overline{n})$, as is easily seen, may be calculated according to the formula

$$\cos\psi = \frac{\operatorname{Re} \sum_{k=1}^{n} u^k \overline{v^k}}{|u| \cdot |v|}, \tag{1.61}$$

where $|u| = (\Sigma_{k=1}^n |u^k|^2)^{1/2}$, $|v| = (\Sigma_{k=1}^n |v^k|^2)^{1/2}$. We take analytic planes U^1, V^1, defined respectively by the vectors u^α, v^α, and consider the minimum of the angles between the vectors αu^k and βv^k lying in these planes (here α and β are arbitrary complex numbers). This minimal value θ is called the angle between the planes U^1, V^1. Further, we choose in V^1 a vector v'^k such that $\angle(u^k, v'^k) = \theta$. Then, if we put $\angle(v^k, v'^k) = \phi$, we have

$$e^{i\phi}\cos\theta = \frac{\sum_{k=1}^{n} u^k \overline{v^k}}{|u|\,|v|};$$

$$\sin\theta = \frac{\left[\sum_{p,q=1,\,p<q}^{n} |u^p v^q - u^q v^p|^2\right]^{\frac{1}{2}}}{|u|\,|v|}. \tag{1.62}$$

The angles θ and ϕ will be called the *first* and *second analytic angles* between the vectors u^k and v^k.

Now we consider in the space C^2 of variables z_1, z_2 an analytic surface element defined by the equation $z_2 = f(z_1)$. Let $M(z_1, z_2)$ and $M'(z_1 + dz_1, z_2 + dz_2)$

be two infinitely close points of this element and $ds = (|dz_1|^2 + |dz_2|^2)^{1/2}$ the distance between them. Let $d\theta$ be the angle between the tangent planes to the surfaces $z_2 = f(z_1)$ at that point, calculated by the second formula (1.62). Then a direct calculation leads us to the equality

$$\frac{d\theta}{ds} = \frac{|f''|}{(1 + |f'|^2)^{\frac{3}{2}}}. \tag{1.63}$$

It is natural to call the quantity (1.63) the *curvature* [1] of the analytic surface $z_2 = f(z_1)$. We note further that an element of an analytic surface is characterized by the fact that the area contained inside some contour lying on the element is less than the area of all other surfaces bounded by the same contour. [2] Hence it follows that the mean curvature of such a surface is equal to zero at all of its points.

§5. EXTENSION OF A SPACE. CONCEPT OF HOLOMORPHIC FUNCTION AT THE POINTS AT INFINITY OF A SPACE

1. Extended plane of one complex variable. As is known, the completion of the plane C^1 of the complex variable z by the point at infinity is carried out by means of stereographic projection. This is done by a mapping of the unit sphere $Q_2\{(x^1)^2 + (x^2)^2 + t^2 = 1\}$ of the space R_3 of real variables x^1, x^2, t onto the plane C^1.

On the plane C^1 we introduce homogeneous coordinates ζ_1, ζ_2, putting

$$z = \frac{\zeta_1}{\zeta_2}, \text{ where } |\zeta_1|^2 + |\zeta_2|^2 \neq 0. \tag{1.64}$$

Each pair of values of the homogeneous coordinates defines some point on the plane C^1, with the exception of the values $\zeta_1 \neq 0$, $\zeta_2 = 0$, to which there

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1) See Fuks [1]. The first expression $\frac{|f''|}{(1 + |f'|^2)^{3/2}}$, as characterizing the curvature of the surface, was considered by K. Kommerell [1]. However the geometric meaning of this expression indicated here was not explained by Kommerell.

2) See Kommerell [1].

correspond no points of the plane C^1.

In homogeneous coordinates the stereographic projection is given by the equations

$$x^1 + ix^2 = x = \frac{1}{N} 2\zeta_1 \overline{\zeta}_2 = \frac{2z}{|z|^2 + 1},$$

$$t = \frac{1}{N}(|\zeta_1|^2 - |\zeta_2|^2) = \frac{|z|^2 - 1}{|z|^2 + 1}, \tag{1.65}$$

where $N = |\zeta_1|^2 + |\zeta_2|^2$. The last terms of (1.65) are admissible only for $\zeta_2 \neq 0$.

As one sees from equation (1.65), to the values $\zeta_1 \neq 0$, $\zeta_2 = 0$ of the homogeneous coordinates there corresponds on the sphere Q_2 its north pole, the point (0, 0, 1). On the other hand, no point of the plane C^1 corresponds to these values of the coordinates ζ_1, ζ_2. Then one introduces a new complex number $z = \infty$ considered as the coordinate (affix) of the point (0, 0, 1) of the sphere Q_2.

The image of the plane C^1 of complex numbers z under (1.65) is the sphere Q_2 with the exception of the point (0, 0, 1). The introduction of the new number $z = \infty$ extends that image to a complete set, namely, the full sphere Q_2. This is the so-called Riemann sphere of the complex variable z.

Then one introduces the concept of functions holomorphic at the point $z = \infty$. The function $f(Z)$ is said to be holomorphic at the point $Z = \infty$ if the function $f((az + b)/(cz + d))$ (where a, b, c, d are numbers taken so that $c \neq 0$, $ad - bc \neq 0$) is holomorphic at the point $z = -d/c$. Here the particular choice of the numbers a, b, c, d is of no importance, since the projective transformations

$$Z = \begin{cases} \dfrac{az + b}{cz + d} & \text{for } z \neq -\dfrac{d}{c}, \\ \infty & \text{for } z = -\dfrac{d}{c} \end{cases} \tag{1.66}$$

form a group of conformal mappings of the Riemann sphere Q_2 onto itself.

Now we consider a bundle π^1 of analytic planes $\Omega = \{\zeta_1 Z_2 - \zeta_2 Z_1 = 0\}$, passing through the origin of coordinates in the space C^2 of variables Z_1, Z_2. In this case we introduce the following topology. By an ϵ-neighborhood of the plane $\Omega_0 \in \pi^1$ we mean the collection of planes $\Omega \in \pi^1$ for which the angle

$\sphericalangle (\Omega_0, \Omega)$ does not exceed the number $\epsilon > 0$. The bundle π^1 provided with the indicated topology will be called *the complex homogeneous projective space* P^1. The relations (1.65) giving the stereographic projection establish a certain correspondence between the planes Ω of the bundle π^1 (which are defined by their coefficients ζ_1 and ζ_2) and the points of the sphere Q_2. We shall show that it defines a regular imbedding of the space P_1 into the space R_3, i.e., that it is one-to-one and locally regular. This last, in our case, means that

$$\text{Rank} \begin{Vmatrix} \dfrac{\partial x^1}{\partial \eta_1} & \dfrac{\partial x^2}{\partial \eta_1} & \dfrac{\partial t}{\partial \eta_1} \\ \\ \dfrac{\partial x^1}{\partial \eta_2} & \dfrac{\partial x^2}{\partial \eta_2} & \dfrac{\partial t}{\partial \eta_2} \end{Vmatrix} = 2 \tag{1.67}$$

throughout π_1. Here η_1, η_2 are the local coordinates of the planes $\Omega \in \pi^1$. These local coordinates are introduced in the neighborhoods of several planes $\Omega', \Omega'', \cdots \in \pi_1$. Taken together these neighborhoods must exhaust the entire bundle π_1.

To verify that property (1.67) holds for the relation (1.65) it is sufficient for us to introduce the local coordinates η_1, η_2 in neighborhoods of the planes $\Omega' = \{Z_1 = 0\}$ (for them $\zeta_1 = 0$, $\zeta_2 \neq 0$) and $\Omega'' = \{Z_2 = 0\}$ (for them $\zeta_1 \neq 0$, $\zeta_2 = 0$), defined by the conditions $\sphericalangle (\Omega', \Omega) < \pi/3$, $\sphericalangle (\Omega'', \Omega) < \pi/3$ (since $2\pi/3 > \pi/2$, these neighborhoods exhaust the entire bundle π_1). In the first neighborhood we put $\eta_1 + i\eta_2 = \zeta_1 / \zeta_2$, in the second $\eta_1 + i\eta_2 = \zeta_2 / \zeta_1$. After this, equation (1.67) is verified immediately.

One immediately finds that relation (1.65) is univalent, and that its inverse is univalent follows from the fact that equations (1.65) may be solved uniquely for the local coordinates η_1 and η_2 in the corresponding neighborhoods.

Thus we have shown that the relation (1.65) in fact determines a regular imbedding of the space P^1 into the space R_3 in the form of the sphere Q_2.

In order to interpret geometrically the process of completion of the plane by the point at infinity one may use instead of the Riemann sphere Q_2 any other regular imbedding of the space P^1, in particular the space P^1 itself. Thus, in the bundle π^1 we make correspond to the plane $Z_2 = 0$, to which there corresponds no finite number z, the number $z = \infty$.

Consider planes $\Omega_1, \Omega_2 \in \pi^1$, points $\omega_1, \omega_2 \in Q_2$ corresponding to them,

and the complex numbers $z_1 = \zeta_1^{(1)}/\zeta_2^{(1)}$, $z_2 = \zeta_1^{(2)}/\zeta_2^{(2)}$ corresponding to them under equations (1.64), (1.65) (for $\zeta_2^{(1)} = 0$, $z_1 = \infty$; for $\zeta_2^{(2)} = 0$, $z_2 = \infty$). Then, as one finds from an appropriate calculation,

$$\sin(z_1, z_2) = \frac{1}{2}\chi(z_1, z_2) =$$

$$= \frac{|\zeta_1^{(2)}\zeta_2^{(1)} - \zeta_2^{(2)}\zeta_1^{(1)}|}{\sqrt{|\zeta_1^{(1)}|^2 + |\zeta_2^{(1)}|^2}\sqrt{|\zeta_1^{(2)}|^2 + |\zeta_2^{(2)}|^2}} =$$

$$= \frac{|z_2 - z_1|}{\sqrt{|z_1|^2 + 1}\sqrt{|z_2|^2 + 1}} \tag{1.68}$$

(the last expression is admissible only for $z_1, z_2 \neq \infty$). Here $\sin(z_1, z_2)$ is the sine of the angle between the planes Ω_1 and Ω_2 (see formula (1.62)), and $\chi(z_1, z_2)$ is the length of the chord of the sphere Q_2 between the points ω_1 and ω_2. The relation (1.68) establishes an obvious relation between the neighborhoods of the planes $\Omega \in \pi^1_*$ and the points $\omega \in Q_2$.

Following Carathéodory, we call the quantity $\chi(z_1, z_2)$ the *chordal distance*[1] between the points z_1, z_2 (where by these points we mean either the corresponding points of the Riemann sphere Q_2 or the corresponding planes of π^1, or the corresponding elements of some other realization of the space P^1). Agreeing to take $\lim_{n\to\infty} z_n = A$ if $\lim_{n\to\infty} \chi(A, z_n) = 0$, we eliminate the necessity of considering the case of an infinite limit separately from the case of a finite limit (which we would have to do with the usual definition of limit).

2. Extension of the plane of n complex variables. We introduce in the space C^n of the complex variables $z_1, \cdots, z_n$ homogeneous coordinates $\zeta_1, \cdots, \zeta_n, \zeta_{n+1}$, putting

$$z_k = \frac{\zeta_k}{\zeta_{n+1}}, \tag{1.69}$$

where $k = 1, \cdots, n$; $|\zeta_1|^2 + \cdots + |\zeta_{n+1}|^2 \neq 0$. Each system of values of the homogeneous coordinates defines some point z of the space C^n, with the

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1) See Carathéodory, *Theory of functions*, Vol. 1, Chelsea Publishing Co., New York, 1954; p. 87 ff.

exclusion of the values with $\zeta_{n+1} = 0$, to which there correspond no points of the space C^n.

Equations (1.69) establish a connection between the points $z \in C^n$ and the bundle π^n of complex homogeneous planes $\Omega = \{Z_{n+1}\zeta_k - Z_k\zeta_{n+1} = 0,\ k = 1, \cdots, n\}$ passing through the origin of coordinates in the space C^{n+1} of variables $Z_1, \cdots, Z_{n+1}$. In addition there are no points $z \in C^n$ corresponding to the planes Ω lying in the n-complex-dimensional space $Z_{n+1} = 0$ (for them $\zeta_{n+1} = 0$).

We establish also a correspondence between the points $z \in C^n$ and the points ω of some manifold Q_{2n} of the space $R_{n(n+2)}$ of real variables x^1_{pq}, x^2_{pq} $(p, q = 1, \cdots, n+1,\ p < q)$, t_k $(k = 1, \cdots, n)$ by means of the equations [1]

$$x_{pq} = x^{(1)}_{pq} + ix^{(2)}_{pq} = \frac{1}{N}\sqrt{\frac{2(n+1)}{n}}\,\zeta_p\bar{\zeta}_q,$$

$$p,\ q = 1, \cdots, n+1\,(p < q),$$

$$t_k = \frac{1}{N}\sqrt{\frac{n+1}{nk(k+1)}}\left[\sum_{s=1}^{k}|\zeta_s|^2 - k|\zeta_{k+1}|^2\right],\quad k = 1, \cdots, n, \tag{1.70}$$

where $N = \sum_{s=1}^{n+1}|\zeta_s|^2$. Here there are no points $z \in C^n$ corresponding to the points ω of the manifold Q_{2n} for which $\zeta_{n+1} = 0$.

Formulas (1.70) for $n = 1$ come down to the formulas (1.65). We have also the following theorems, which are verified immediately on computation.

THEOREM 5.1. *The manifold Q_{2n} lies on the unit sphere*

$$\sum_{\substack{p,q=1\\ p<q}}^{n+1}|x_{pq}|^2 + \sum_{k=1}^{n}|t_k|^2 = 1. \tag{1.71}$$

THEOREM 5.2. *If the planes $\Omega_1, \Omega_2 \in \pi^n$ and the points $\omega_1, \omega_2 \in Q_{2n}$ are defined respectively by the homogeneous coordinates $\zeta^{(1)}_1, \cdots, \zeta^{(1)}_{n+1}$ and $\zeta^{(2)}_1, \cdots, \zeta^{(2)}_{n+1}$ (the coordinates of the points ω_1 and ω_2 in the space $R_{n(n+2)}$ are found from equations* (1.70), *then*

1) See Fuks [5].

$$\sin(\Omega_1, \Omega_2) = \sqrt{\frac{n}{2(n+1)}}\, \chi(\omega_1, \omega_2) = \\ = \frac{1}{\sqrt{N_1 N_2}} \left[\sum_{\substack{p, q = 1, \cdots, n+1 \\ p<q}} | \zeta_p^{(1)} \zeta_q^{(2)} - \zeta_p^{(2)} \zeta_q^{(1)} |^2 \right]^{\frac{1}{2}}, \tag{1.72}$$

where $N_k = \Sigma_{s=1}^{n+1} | \zeta_s^{(k)} |^2$, $k = 1, 2$.

Here $\chi(\omega_1, \omega_2)$ is the distance between the points ω_1 and ω_2 in the space $R_{n(n+2)}$, $\sin(\Omega_1, \Omega_2)$ the sine of the angle between the planes Ω_1 and Ω_2. From this theorem it follows in particular that the diameter of the manifold Q_{2n} is equal to $\sqrt{2(n+1)/n}$.

Let us introduce in the bundle π^n the following topology. By an ϵ-neighborhood of the plane $\Omega_0 \in \pi^n$ we mean the collection of planes $\Omega \in \pi^n$ for which the angle $\measuredangle(\Omega_0, \Omega)$ does not exceed the number $\epsilon > 0$. The bundle π^n provided with the indicated topology will be called a *complex n-dimensional projective space.*

THEOREM 5.3. *The equations* (1.70) *define a regular imbedding of the space* P^n *into the space* $R_{n(n+2)}$.

The fact that the relation (1.70) has the properties expressed by the theorems above leads us to call it an n-complex-dimensional stereographic projection.

In considering relations (1.69) and (1.70) we shall call the points of the space P^n (i.e., the planes $\omega \in \pi^n$ or the points $\omega \in Q_{2n}$) *finite* if points $z \in C^n$ correspond to them, and *points at infinity* if no points $z \in C^n$ correspond to them. The passage from the space C^n to the space P^n is usually called an *extension* of the space C^n. As we see, this transition consists in complementing the image of the space C^n in the space P^n by points at infinity. We note that P^n, in distinction from C^n, is complete.

The collection of points $\zeta \in P^n$ whose homogeneous coordinates $\zeta_1, \cdots, \zeta_{n+1}$ satisfy the equations

$$\sum_{s=1}^{n+1} a_{ks} \zeta_s = 0, \quad k = 1, \cdots, n-m, \tag{1.73}$$

where $\operatorname{Rank} \| a_{ks} \| = n - m$, will be called an m-complex-dimensional complex projective space P^m. Such a plane is itself an m-complex-dimensional complex projective space. We note that the points at infinity of the space P^n (defined by the equation $\zeta_{n+1} = 0$) constitute an $(n-1)$-complex-dimensional complex

projective plane. The points at infinity belonging to the plane P^m that is defined by the equations

$$\sum_{s=1}^{n} a_{ks}\zeta_s - b_k\zeta_{n+1} = 0, \quad k = 1, \cdots, m, \tag{1.74}$$

where $\operatorname{Rank}\| a_{ks} \| = n - m$ (they form an $(m-1)$-complex-dimensional complex projective plane), will also be called the points at infinity of the analytic plane C^m defined by the equations

$$\sum_{s=1}^{n} a_{ks} z_s - b_k = 0, \quad k = 1, \cdots, m. \tag{1.75}$$

From what has been said it follows in particular that the plane P^1 or C^1 always has one point at infinity. Therefore the points at infinity of the space P^n may be given by indicating the planes $\Omega\{z_1 = 0, \cdots, z_{\nu-1} = 0,\ z_{\nu+1} = \alpha_{\nu+1} z_\nu, \cdots$ $\cdots, z_n = \alpha_n z_\nu\}$ to which they belong. Here $\{\Omega\}$ is the bundle of one-complex-dimensional analytic planes with vertex at the origin of coordinates, and z_ν is the coordinate of lowest index taking nonzero values in the plane Ω. The point at infinity of such a plane will be indicated by means of its homogeneous coordinates or by the symbol $(\underbrace{0, \cdots, 0}_{\nu-1 \text{ times}}, \infty, \alpha_{\nu+1}, \cdots, \alpha_n)$.

The regions of the space P^n containing the points at infinity are also called *infinite*. Regions not containing any points at infinity are said to be *finite*.

If the points $\zeta^{(1)}, \zeta^{(2)} \in P^n$ and ω_1, ω_2 are the points of Q_{2n}, then the quantity $\chi(\zeta^{(1)}, \zeta^{(2)}) = \chi(\omega_1, \omega_2)$ defined by equation (1.72) will be said to be the *chordal distance* between the points $\zeta^{(1)}, \zeta^{(2)}$. If $\zeta^{(n)}$, where $n = 1, 2, \cdots$, is a sequence of points of the space P^n and the point $A \in P^n$, then we will say that $\lim_{n\to\infty} \zeta^{(n)} = A$ if $\lim_{n\to\infty} \chi(A, \zeta^{(n)}) = 0$. Suppose that we are given a function $w = f(\zeta)$, where $\zeta \in P^n$, $w \in P^1$, defined in some neighborhood of the point $\zeta^{(0)} \in P^n$, with the possible exclusion of the point $\zeta^{(0)}$ itself. We will say that $\lim_{\zeta\to\zeta^{(0)}} w(\zeta) = w_0$, where $w_0 \in P^1$, if for each $\epsilon > 0$ we can find a number $\delta = \delta(\epsilon) > 0$, such that for $0 < \chi(\zeta, \zeta_0) < \delta(\epsilon)$ we have $\chi(w, w_0) < \epsilon$. Analogously, using the chordal distance, we define the concept of a function continuous at some point $\zeta^{(0)} \in P^n$, and of a function uniformly continuous in some region $\Delta \subset P^n$. As we have already indicated in the case $n = 1$, these definitions avoid the necessity of separate consideration

of infinite limits and limits obtained on approach to points at infinity. We must only mention that the Weierstrass Theorems 1.2 and 1.3 on functions continuous in a closed region remain valid only for finite continuous functions. The definition of uniform convergence for the space P^n remains the same as in the space C^n.

3. Concept of functions holomorphic at points of the space P^n. Finite points of the space P^n, which contains as part of itself the image of the space C^n, may be considered simultaneously as points of both spaces. Therefore we may give the following definition.

DEFINITION (holomorphicity at a finite point). The function $w = f(\zeta)$ (where $w \in C^1$, $\zeta \in P^n$) is said to be holomorphic at the finite point $\zeta(\zeta_1, \cdots, \zeta_{n+1}) \in P^n$ if the function is holomorphic at the point $z(z_1, \cdots$ $\cdots, z_n) \in C^n$ for which the numbers $\zeta_1, \cdots, \zeta_{n+1}$ are homogeneous coordinates.

For the definition of the concept of a function holomorphic at the point at infinity $\eta(\eta_1, \cdots, \eta_{n+1}) \in P^n$, we consider the projective mapping $\omega = A\zeta$, defined by the equations

$$\omega_k = \sum_{s=1}^{n+1} a_{ks} \zeta_s, \tag{1.76}$$

where $k = 1, \cdots, n+1$, $\operatorname{Det} a_{ks} \neq 0$ and the points $\zeta(\zeta_1, \cdots, \zeta_{n+1})$, $\omega(\omega_1, \cdots$ $\cdots, \omega_{n+1}) \in P^n$. Under an appropriate choice of the coefficients a_{ks} in this mapping we may obtain the point η as the image of some finite point $\xi(\xi_1, \cdots$ $\cdots, \xi_{n+1}) \in P^n$. This makes it possible to give the following definition.

DEFINITION (holomorphicity at a point at infinity). The function $w = f(\zeta)$ (where $\zeta \in P^n$, $w \in C^1$) is said to be holomorphic at the point at infinity $\eta \in P^n$ if the function $f(A\zeta)$ is holomorphic at the finite point $\xi \in P^n$. Here $\eta \in A\xi$, A the mapping (1.76).

The projective transformations (1.76) constitute a group of one-to-one mappings of the space P^n onto itself and are expressed by holomorphic functions. Therefore, by the theorem on holomorphicity of a composite function, we may assert that the arbitrariness in the choice of the finite point $\xi \in P^n$ in our definition is not essential. Usually in the verification of the holomorphicity of a function f at a point at infinity η the transformations (1.76) are given the simplest possible form. Thus one may state that holomorphicity of the function f at the point at infinity $\eta(0, \cdots, 0, \infty, a_{\nu+1}, \cdots, a_n)$ is equivalent to the holomorphicity of the function

$$f\left(\frac{z_1}{z_\nu},\cdots,\frac{z_{\nu-1}}{z_\nu},\frac{1}{z_\nu},\frac{z_{\nu+1}+\alpha_{\nu+1}}{z_\nu},\cdots,\frac{z_n+\alpha_n}{z_\nu}\right)$$

at the origin of coordinates of the space C^n. In order to obtain this result it is sufficient to replace the general projective mapping (1.76) by a projective mapping of special form:

$$\begin{aligned}&\omega_k=\zeta_k \text{ (for } k=1,\cdots,\nu-1),\ \omega_\nu=\zeta_{n+1},\\ &\omega_k=\zeta_k+\alpha_k\zeta_{n+1} \text{ (for } k=\nu+1,\cdots,n),\ \omega_{n+1}=\zeta_\nu.\end{aligned} \tag{1.76$_1$}$$

We leave to the reader the verification that the point $\eta(0,\cdots,0,\infty,\alpha_{\nu+1},\cdots$ $\cdots,\alpha_n)$ is the image of the origin of coordinates of the space C^n of variables $z_1,\cdots,z_n$ under that mapping.

One may express the condition of holomorphicity of a function f at the point at infinity $\eta(0,\cdots,0,\infty,\alpha_{\nu+1},\cdots,\alpha_n)$ by means of power series. Using the transformation inverse to (1.76$_1$), we deduce that the function $f(\omega)$ will be holomorphic at the point η if and only if the equation

$$f(\omega)=\sum_{k_1,\cdots,k_n=0}^{\infty}a_{k_1\cdots k_n}\left(\frac{\omega_1}{\omega_\nu}\right)^{k_1}\cdots\left(\frac{\omega_{\nu-1}}{\omega_\nu}\right)^{k_{\nu-1}}\left(\frac{\omega_{n+1}}{\omega_\nu}\right)^{k_\nu}\times$$

$$\times\left(\frac{\omega_{\nu+1}-\alpha_{\nu+1}\omega_\nu}{\omega_\nu}\right)^{k_{\nu+1}}\cdots\left(\frac{\omega_n-\alpha_n\omega_\nu}{\omega_\nu}\right)^{k_n} \tag{1.77}$$

holds for the points $\omega\in S_R$, where S_R is the region of the space P^n defined by the conditions

$$(S_R)\quad \begin{aligned}&\left|\frac{\omega_1}{\omega_\nu}\right|<R,\cdots,\left|\frac{\omega_{\nu-1}}{\omega_\nu}\right|<R,\ \left|\frac{\omega_{n+1}}{\omega_\nu}\right|<R,\\ &\left|\frac{\omega_{\nu+1}}{\omega_\nu}-\alpha_{\nu+1}\right|<R,\cdots,\left|\frac{\omega_n}{\omega_\nu}-\alpha_n\right|<R.\end{aligned} \tag{1.78}$$

Here R is some positive number. We note that in the region S_R there will lie the finite points of the space $C^n\subset P^n$ whose coordinates satisfy the conditions (here $z_k=\omega_k/\omega_{n+1}$; $k=1,\cdots,n$)

$$(S_R \cap C^n)\quad \begin{array}{l} \left|\dfrac{z_1}{z_\nu}\right| < R, \cdots, \left|\dfrac{z_{\nu-1}}{z_\nu}\right| < R, \ |z_\nu| > R, \\ \left|\dfrac{z_{\nu+1}}{z_\nu} - \alpha_{\nu+1}\right| < R, \cdots, \left|\dfrac{z_n}{z_\nu} - \alpha_n\right| < R. \end{array} \tag{1.78'}$$

The region S_R in which the series (1.77) converges for the largest possible R will be called an elementary region of convergence of that series. It is easy to see that on the boundary of an elementary region of convergence of the series (1.77) there are necessarily points at which the holomorphicity of its sum is destroyed.

Functions holomorphic at some point at infinity of the space P^n are bounded in some neighborhood of this point. This circumstance makes it possible to formulate the theorem of Liouville given in 3.12 as follows:

Functions holomorphic at all the points of a complex projective space P^n are constant.

The functions holomorphic at some point $\zeta \in P^n$ form a ring of integrity $\mathfrak{O}_\zeta$. Theorems 4.5 and 4.8 are valid for them. Those functions which are holomorphic in some region $D \subset P^n$ constitute a ring of integrity. Functions holomorphic on some open set $B \subset P^n$ also form a commutative ring. However it generally speaking is not an integral domain.

The collection of rings $\mathfrak{O}_\zeta$ for various points $\zeta \in D$ or $\zeta \in B$ constitute a bundle, which is denoted by the symbol $\mathfrak{O}(D)$ or $\mathfrak{O}(B)$.

Analogous remarks may be made concerning the collection of functions holomorphic at the points of the space G^n of the theory of functions. This space will be considered in the following subsection.

4. The space of the theory of functions. The method of extending the space C^n described above is not the only possible one. In the theory of functions there is still a further way of extending that space, which we shall now present.

We consider the plane C_k^1 of the variable z_k and the method indicated above of completing it to the space P_k^1. Then we form the product of these spaces $P_1^1 \times \cdots \times P_n^1$. As is known, we may think of each space P_k^1 as a sphere $Q_2^{(k)}$ constructed for the corresponding variable z_k. The product of these spheres is a complete set containing as a subset the space C^n of the variables $z_1, \cdots, z_n$. More precisely, it contains a subset which is the

product of spheres $Q_2^{(k)}$ punctured in the points $z_k = \infty$ and homeomorphic to C^n. We will call the space $P_1^1 \times \cdots \times P_n^1$ *the space of the theory of functions* G^n. In it each complex-homogeneous analytic plane $z_k = z_k^0$ $(k \neq \nu)$ has one point at infinity, which we denote by the symbol $(z_1^0, \cdots, z_{\nu-1}^0, \infty, z_{\nu+1}^0, \cdots$ $\cdots, z_n^0)$. Each two-complex-dimensional analytic plane $z_k = z_k^0$ $(k \neq \nu, \mu)$ has, aside from those indicated above, still another point at infinity which for $\nu = 1$, $\mu = 2$ we shall indicate by the symbol $(\infty, \infty, z_3^0, \cdots, z_n^0)$ and so forth.

In order to extend the concept of a holomorphic function to the points at infinity of the space G^n we make use of the transformations (1.66) onto the sphere $Q_2^{(k)}$ of each variable z_k. As a result we arrive at the following definition.

DEFINITION (holomorphicity at a point at infinity of the space G^n). The functions $w = f(\zeta)$ (where $\zeta \in G^n$, $w \in C^1$) are said to be holomorphic at the point at infinity $\eta \in G^n$ if the function

$$f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \cdots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right)$$

(where $a_k d_k - c_k b_k \neq 0$ for all $k = 1, \cdots, n$) is holomorphic at the corresponding finite point $z(z_1, \cdots, z_n) \in C^n$.

At the indicated point all or part of the denominators $c_k z_k + d_k$ turn out to be equal to zero. We observe also that here too it is not necessary to consider the most general transformations (1.66). Just as in the preceding case, here one may restrict oneself to transformations of a more special sort. We shall make use of this fact in the following section.

5. Laurent series. In order to preserve the notation we consider the case of two variables w, z. To study the behavior of the function $f(w, z)$ at the points at infinity it is sufficient to consider transformations of the form (1.66) with

$$W = w, \quad Z = \frac{1}{z}; \quad W = \frac{1}{w}, \quad Z = z;$$
$$W = \frac{1}{w}, \quad Z = \frac{1}{z}, \tag{1.79}$$

carrying respectively the points $(a, 0)$, $(0, a)$ and $(0, 0)$ into the points (a, ∞), (∞, a) and (∞, ∞) of the space G^2. In view of what has been said the function $f(w, z)$ is holomorphic at these points if the functions $f(w, 1/z)$, $f(1/w, z)$,

$f(1/w,\ 1/z)$ are holomorphic respectively at the points $(a, 0)$, $(0, a)$, $(0, 0)$.

Applying the transformations (1.79) to the expansion of the functions $f(w, z)$ in double Taylor series in the neighborhood of the points $(a, 0)$, $(0, a)$, $(0, 0)$, we may formulate the conditions of holomorphicity of the function at points at infinity also in the following way.

The expression "the function $f(w, z)$ is holomorphic at the point (a, ∞)" means that there exists a bicylindrical region [1] $\{|w-a| < R,\ |z| > 1/R\}$ in which $f(w, z)$ is represented by the series

$$f(w, z) = \sum_{k,\, l=0}^{\infty} c_{kl}(w-a)^k \frac{1}{z^l}. \tag{1.80}$$

Analogously for the points (∞, a) and (∞, ∞) one has respectively to consider the bicylindrical regions $\{|w| > 1/R,\ |z-a| < R\}$, $\{|w| > 1/R,\ |z| > 1/R\}$ and for them the series

$$\left.\begin{aligned} f(w, z) &= \sum_{k,\, l=0}^{\infty} c_{kl} \frac{1}{w^k} (z-a)^l; \\ f(w, z) &= \sum_{k,\, l=0}^{\infty} c_{kl} \frac{1}{w^k} \frac{1}{z^l}. \end{aligned}\right\} \tag{1.80'}$$

We note that the series (1.80) and those analogous to it, just as in the corresponding expansion in the classical theory of functions, appear as part of the expansion of Laurent type.

A *Laurent expansion* in our case may be obtained as follows. Suppose that the function $f(w, z)$ is holomorphic at the points of the bicylindrical region $D = D_1 \times D_2$, where D_1, D_2 are annular rings in the planes of w and z defined by the conditions $r_1 < |w| < R_1$, $r_2 < |z| < R_2$, and is continuous in the closed region $\overline{D}$. If C_1 and C_2 are the circles bounding the ring D_1, and Γ_1, Γ_2 are the circles bounding the ring D_2, then for each point (w, z) of the region D we will have

1) This region forms an elementary neighborhood of the point at infinity for the largest possible value of R.

$$f(w, z) = -\frac{1}{4\pi^2} \int_{C_1} \frac{dt_1}{t_1 - w} \int_{\Gamma_1} \frac{f(t_1, t_2)}{t_2 - z}\, dt_2 -$$

$$-\frac{1}{4\pi^2} \int_{C_1} \frac{dt_1}{t_1 - w} \int_{\Gamma_2} \frac{f(t_1, t_2)}{t_2 - z}\, dt_2 - \frac{1}{4\pi^2} \int_{C_2} \frac{dt_1}{t_1 - w} \int_{\Gamma_1} \frac{f(t_1, t_2)}{t_2 - z}\, dt_2 -$$

$$-\frac{1}{4\pi^2} \int_{C_2} \frac{dt_1}{t_1 - w} \int_{\Gamma_2} \frac{f(t_1, t_2)}{t_2 - z}\, dt_2$$

(the direction of traversing the circles $C_1, C_2, \Gamma_1, \Gamma_2$ is such that we pass around the areas of the annuli D_1, D_2 in the same way as around the triangle with the vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and with the indicated order of succession of its vertices). The integrals obtained by the usual method lead us to series with positive and negative powers of w and z, and we obtain the Laurent expansion of the function $f(w, z)$ in the region D:

$$f(w, z) = \sum_{k, l=-\infty}^{\infty} c_{kl} w^k z^l. \tag{1.81}$$

The series (1.81) may be considered as the sum of the expansions (1.36) and (1.80) and of expansions analogous to them, representing $f(w, z)$ in the common part of the regions of convergence of these series.

Along with the space of the theory of functions $G = \underbrace{P^1 \times \cdots \times P^1}_{n \text{ times}}$ we may consider also spaces of the more general form $P^{m_1} \times \cdots \times P^{m_\nu}$, where $\Sigma_\nu m_\nu = n$. For all these spaces we have the Liouville theorem in the formulation presented above for the complex projective space P^n.

§6. ANALYTIC CONTINUATION OF FUNCTIONS AND SETS

1. Analytic continuation. Suppose that D and D_0 are regions of the space P^n or G^n, with $D_0 \subset D$. If f_0 and f are holomorphic functional elements in the regions D_0 and D, while $f_0 = f$ for the points of the region D_0, then the function f is said to be an *analytic continuation of the function f_0 from the region D_0 to the region D.*

Suppose that the regions D_0 and D_1 intersect in some region G_0 and that in the region $D = D_0 \bigcup D_1$ there exists a holomorphic function f which is a

continuation of the function f_0 given in the region D_0 and simultaneously a continuation of the function f_1 given in the region D_1. Then the function f_1 is said to be an *immediate analytic continuation of the function* f_0 *into the region* D_1. In view of the uniqueness theorem an immediate analytic continuation is always unique.

Now we consider regions $D_0, D_1, \cdots, D_m$. Suppose that the regions D_k and D_{k+1} intersect in a nonempty region G_k $(k = 0, 1, \cdots, m-1)$. Suppose that $f_0, \cdots, f_m$ are holomorphic functional elements given in the respective regions $D_0, \cdots, D_m$, while the element f_{k+1} is an immediate analytic continuation of the element f_k to the region D_{k+1}. In this case we will call the function f_m *an analytic continuation of the holomorphic functional element* f_0 *to the region* D_m, the elements f_0 and f_m *joined*, the elements f_k and f_{k+1} (for all $k = 0, 1, \cdots, m-1$) *adjacent*. One must at once realize that as a result of changing even a portion of the intervening regions $D_1, \cdots, D_{m-1}$ we may obtain in D_m as the analytic continuation of the element f_0 a holomorphic functional element different from f_m. Thus, the unique character of the immediate analytic continuation is lost.

2. Analytic continuation along a curve. Suppose that L is some piecewise smooth curve in the space P^n or G^n, joining two points M_0 and M of that space. Suppose that f_0 is a holomorphic functional element with center at the point M_0 and S_0 is an elementary neighborhood of convergence for the power series with center at the point which represents this function (in the case of a point at infinity one considers instead the series (1.77), (1.80)). The point M_1 is so chosen on the curve L inside the region S_0 that the segment M_0M_1 of the curve L entirely lies in that region.

By transforming the power series representing the function f_0 into a new one with center at the point M_1, we define a holomorphic functional element f_1. The element f_1 is an immediate analytic continuation of the holomorphic functional element f_0. If it turns out to be possible to choose the point M_1, preserving the conditions indicated above, in such a way that inside its elementary neighborhood S_1 there appear points of our curve not belonging to the region S_0 and forming with the points inside the region S_0 a connected portion of the curve L, then we will have taken an effective step in the analytic continuation of the element f_0 along the curve L. Next we choose in the region S_1 on the curve L a point M_2 (of course, not on the segment M_0M_1), and then

transform the series representing the function f_1 so as to define a holomorphic functional element f_2 with center at the point M_2 and in a corresponding elementary neighborhood S_2. Here the segment $M_1 M_2$ of the curve L must lie entirely within the region S_1. In the case when as a result of a finite set of steps we obtain holomorphic functional elements $f_0, \cdots, f_m$ with centers at the points $M_0, \cdots, M_m = M$ of the curve L, then we say that in this way the holomorphic functional element f_0 has been continued from the point M_0 to the point M along the curve L and the function $f = f_m$ will be called *the analytic continuation* of the element f_0 along the curve.

Note that because of the uniqueness of the immediate analytic continuation, analytic continuation along the curve L is unique and does not depend on the choice of the points $M_1, \cdots, M_{m-1}$. If as intermediate points the points $M_1', \cdots, M_{m-1}'$ of the curve L were chosen, then at the point M we would obtain the same functional element f. This may be proved in the same way as in the case of one variable, by considering the system of points which is formed by the systems of points M_k and M_k' in the order in which they lie along the curve between the points M_0 and M.

If the points M_0 and M are joined by two different paths L and L', then the results of analytic continuation of the function f_0 to the point M along the paths L and L' turn out in general to be different.

One may however, as in the case of one variable, show that the curve L may be surrounded by a region — a neighborhood [1] $\tilde{L}$ — such that as a result of continuation of the element f_0 along each path $M_0 M$ lying in the region $\tilde{L}$ we obtain at the point M one and the same functional element.

More generally, two curves L and L' joining the points M_0 and M are said to be equivalent if as a result of analytic continuation along them of the functional element f_0 we obtain at the point M one and the same functional element.

1) Obtained for example as a result of joining the points of the polycylindrical regions $\{|\zeta_k - z_k| < \epsilon;\ k = 1, \cdots, n\}$ attached to each point $R(z_1, \cdots, z_n)$ of the curve L for an appropriate choice of the number ϵ. If A is the point at infinity of the space P^n, then to it one attaches a region defined by the condition $\chi(\zeta, A) < \epsilon$, where χ is the chordal distance, or by conditions of the form (1.78). The corresponding regions for the space G^n are indicated in subsection 5 of § 5.

From the foregoing it is furthermore easy to derive the following assertion.

THEOREM 6.1 (monodromy). *If the functional element f can be analytically continued in the simply-connected region D from one of its points along every curve lying entirely in D, then the function constructed in this way will be uniquely defined in the region D (i.e., if we continue it along any closed curve L lying entirely in the region D, we arrive at its initial value).*

PROOF. Let P be a point at which the functional element f is given, and Q any other point of the region D. Let L and L' be two curves joining the points P and Q and lying entirely in the region D. Since the region D is simply-connected, there exists a family of curves $\{L_t, 0 \leq t \leq 1, L_0 = L, L_1 = L'\}$ continuously depending on the parameter t (i.e., $\lim_{\Delta t \to 0} r(L_t, L_{t+\Delta t}) = 0$, where r is the distance between the curves). We denote by τ the upper bound of the values of t for which the functional elements obtained at the point Q by analytic continuation of the functional element f along the paths L_0 and L_t coincide. The path L_τ evidently also has this property. From what has been said above, the same result must hold also for all the paths lying in some neighborhood U of the curve L_τ. If $\tau < 1$, then there is a number $\epsilon > 0$ such that for $0 \leq \alpha < \epsilon$ all $L_{t+\alpha} \subset U$. But in this case the number τ is not an upper bound for the numbers t as assumed. Accordingly $\tau = 1$ and our assertion is proved.

In §2 we have established that if the function $f(z)$ is holomorphic in the region D and if the piecewise smooth curve $L \subset D$ and the points $z, z^0 \in L$, then

$$f(z) = f(z^0) + \int_{z^0}^{z}{}_L \, df.$$

In view of the uniqueness of the functional element f in the region D, we may consider its value as having been obtained by analytic continuation of the function f from the point z^0 to the point z along the curve L (within the limits of the region D).

However on changing the path L we in general obtain a different result. But for a simply-connected region D, from the monodromy theorem just proved, it follows that the value of f does not depend on the path L and hence is uniquely defined. In this case for each closed path

$$\oint df = 0.$$

Conversely, the monodromy theorem can be proved by starting from the fact that the curvilinear integral

$$\oint df$$

in a simply-connected region does not depend on the path of integration (since the appropriate conditions on its real and imaginary parts are satisfied).

REMARK. The monodromy theorem and the fact that the integral is independent of the path of integration may be established for regions of more general character. They are true for regions in which every closed contour is only homologous to zero (and not homotopic to zero, as required in our definition of simply-connected region). One arrives at this conclusion by considering the integrals along the edges of simplexes into which the surface spanning the arbitrary contour has been subdivided by triangulation.

3. Concept of complete analytic function. Let f_0 be a holomorphic functional element at the point M_0. The point M is said to be *accessible* if there exists a path L along which the element f_0 can be continued from the point M_0 to the point M.

It is quite evident that if M is an accessible point, then as a result of continuation of the element f_0 to the point M along all possible nonequivalent paths we will obtain at the point M itself an at most countable set of functional elements. As in the case of one variable, this fact is easily proved if by means of a small deformation of the curve L (by the above, this will not change the values of the function f at the point M) we arrange matters so that all the intermediate points M_k have rational coordinates.

Consider a region D consisting of points which are accessible in analytic continuation of the holomorphic functional element f_0. The collection of holomorphic elements obtained at the points of the region D as a result of this continuation is said to be *an analytic function* of the variables $z_1, \cdots, z_n$ in the region D. It is a *section* of the bundle $\mathfrak{O}(D)$. The holomorphic elements belonging to some analytic function are frequently called its *holomorphic branches*. An analytic function generally speaking is multiple-valued.

If the region D is the set of all admissible points and if one takes all the holomorphic elements arising from continuation along all nonequivalent paths, the analytic function so obtained is said to be *complete*.

4. Continuation of analytic sets and surfaces. Consider some analytic subset m of the open set $B \subset C^n$. If $\widetilde{m}$ is an analytic subset of the open set

$\widetilde{B} \supset B$, while $\widetilde{m} \cap B = m$, $\widetilde{m} \neq m$, then the analytic set $\widetilde{m}$ is said to be a continuation of the analytic set m to the open set $\widetilde{B}$. If the open set $B_1 \subset \widetilde{B}$, then the analytic set $\widetilde{m} \cap B_1$ in the case at hand is also called the continuation of the analytic set m to the open set B_1. If we take various open sets $\widetilde{B}$ containing the sets B and B_1, we will obtain different continuations, generally speaking, of the analytic set m to the open set B_1.

An analytic set $M \subset C^n$ which does not admit a continuation is said to be *complete*, its boundary points are called its *singular* points, and the analytic sets $m \subset M$ are its *elements*.

The concept of continuation applies also to analytic surfaces of any dimensionality. An analytic surface which cannot be continued as an analytic surface is said to be a *complete analytic surface*. We observe that among the boundary points of such a complete surface we must also include the points of intersection of its various elements obtained in the process of continuation (in the neighborhood of such a point a complete analytic surface cannot be an *irreducible* analytic surface).

DEFINITIONS. 1 (selfintersection point). A boundary point P of a complete analytic surface is called a *selfintersection point* of that surface if in some neighborhood of the point P, including the boundary points which may belong to that neighborhood, the surface is a reducible analytic set. 2 (singular point). The remaining boundary points of a complete analytic surface are called *singular points* of that surface.

EXAMPLE. Consider an element of an analytic surface, defined in the neighborhood of the point $(0, 0)$ by the equations $w = z\sqrt{1 + z}$ (where $\sqrt{1} = +1$). As a result of continuation our element enters into the structure of the complete analytic surface $w^2 - z^2(1 + z) = 0$. In this case there passes through the point $(0, 0)$ another element of the surface defined by the equation $w = - z\sqrt{1+x}$ (here we take $\sqrt{1} = + 1$). The collection of all the points of this complete analytic surface in the neighborhood of the point $(0, 0)$ is defined by an equation reducible at the point $(0, 0)$, since

$$w^2 - z^2(1 + z) = (w - z\sqrt{1 + z})\,(w + z\sqrt{1 + z}).$$

5. Hartogs' theorem on analytic continuation. From the integral theorem of Cauchy for polycylindrical regions and the preparation theorem of Weierstrass one may obtain a series of properties relating to analytic continuation of holomorphic functions. In a series of cases it is possible to establish that all

functions which are holomorphic and bounded in some region $D \subset C^n$ may be analytically continued to a region $D_1 \supset D$. We note that a similar fact is encountered in the theory of functions of a single variable. There one proves that if the function $f(z)$ is holomorphic and bounded in the region of some point $z = z_0$, with the possible exclusion of the point $z = z_0$ itself, then after redefinition or revision of its values at that point it becomes holomorphic in an entire neighborhood of the point z_0. This is the so-called theorem of Riemann on removable singularities of a holomorphic function of one variable.

Furthermore it is possible to establish that in the space C^n (for $n > 1$) there exist regions D with the following property: all the functions holomorphic in such a region may be analytically continued across its limits to some region D_1 containing points lying outside the region D. This fact, as is known, has no analogue in the theory of functions of one complex variable.

The existence of such regions $D \subset C^n$ (for $n > 1$) follows from the theorem proved below.

THEOREM 6.2 (Hartogs [2]). *Suppose that D and E are regions of the w and z planes such that $D \times E$ is a bounded ordinary bicylindrical region, and suppose that the region $K \subset E$. Suppose that the function $f(w, z)$ is given at the points of the set $S\{(\overline{D} \times K) \cup (\partial D \times \overline{E})\}$, and that it is known that it is* 1) *holomorphic in the region $D \times K$ and continuous in w in $\overline{D}$ for any $z \in K$,* 2) *continuous on the skeleton $\partial D \times \partial E$ in the set of its variables,* 3) *holomorphic in E and continuous in $\overline{E}$ in z for any $w \in \partial D$. Then in the region $D \times E$ there exists a holomorphic function $\tilde{f}(w, z)$ whose values at the points of the region $D \times K$ coincide with the values of the given function $f(w, z)$. (Thus, the function $\tilde{f}(w, z)$ is an analytic continuation of the function $f(w, z)$ to the region $D \times E$.)*

PROOF. From hypothesis 1), the function $f(w, z)$ may be represented as follows in the region $D \times K$:

$$f(w, z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\omega, z)}{\omega - w} d\omega.$$

From hypothesis 3), the function $f(\omega, z)$ may be represented as follows in the region E:

$$f(\omega, z) = \frac{1}{2\pi i} \int_{\partial E} \frac{f(\omega, \zeta)}{\zeta - z} d\zeta.$$

Beginning with these representations we obtain for the values of the function

$f(w, z)$ in the region $D \times K$:

$$f(w, z) = -\frac{1}{4\pi^2} \int_{\partial D} \frac{d\omega}{\omega - w} \int_{\partial E} \frac{f(\omega, \zeta)}{\zeta - z} d\zeta =$$

$$= -\frac{1}{4\pi} \int\int_{\partial D \times \partial E} \frac{f(\omega, \zeta)}{(\omega - w)(\zeta - z)} d\omega \wedge d\zeta. \tag{1.82}$$

But from hypothesis 2) this last integral is an integral of Cauchy type and defines a function holomorphic in the entire region $D \times E$. Taking this function as $\tilde{f}(w, z)$, we obtain the result given in the assertion of the theorem.

Complement to Hartogs' theorem. If we replace the requirements 1), 2), 3) in Hartogs' theorem by the condition that the function $f(w, z)$ should be holomorphic at the points of a closed set $\overline{S}$, then one can remove the requirement on piecewise smoothness of the boundaries of the regions D and E. Thus the theorem of Hartogs may be formulated as follows:

THEOREM 6.3. *Suppose that D and E are bounded regions on the planes of w and z, and the region $K \subset E$. If in some neighborhood $\tilde{S}$ of the set $S\{(\overline{D} \times \overline{K}) \cup [\partial D \times \overline{E}]\}$ there is given some holomorphic function $f(w, z)$, then in the region $D \times E$ there exists a holomorphic function $\tilde{f}(w, z)$ whose values at the points of $D \times K$ coincide with the values of the given function $f(w, z)$.*

PROOF. Since the set $\overline{S}$ is closed, one can find an $a > 0$ such that if the point $(w_0, z_0) \in \overline{S}$ then the bicylinder $\{|w - w_0| < a, |z - z_0| < a\} \subset \tilde{S}$ and the function $f(w, z)$ is holomorphic in it. We cover the closed regions $\overline{D}$ and and $\overline{E}$ by squares with sides $a/2$ and denote by D_1 and E_1 the regions formed by all squares having inside or on their boundaries points belonging to $\overline{D}$ or $\overline{E}$. Here the sides of the squares are included in D_1 or E_1 only in the case when both of the squares adjacent to this square belong to that region. Each point of $\overline{D}_1$ (or $\overline{E}_1$) not belonging to D (or E) is distant from D (or E), in particular from the boundary ∂D (or ∂E) by not more than $a\sqrt{2}/2 < a$. Therefore under the conditions of our theorem the regions D, E may be replaced by the regions D_1, E_1. The boundaries of these latter regions are piecewise smooth, since they consist of a finite set of segments of straight lines. Thus our present proposition can be obtained from the basic formulation of Hartogs' theorem.

Theorem 6.2 can easily be extended to the case of any number of variables. In order to avoid excessive complications in the formulation, we present it for the case of three variables.

THEOREM 6.4. *Suppose that* D_k *is a region in the plane of the variables* z_k $(k = 1, 2, 3)$ *such that* $D_1 \times D_2 \times D_3$ *is a bounded ordinary polycylindrical region, and suppose that the regions* $K_k \subset D_k$ $(k = 2, 3)$. *Suppose that the function* $f(z_1, z_2, z_3)$ *is given at the points of the set* $S\{(\overline{D}_1 \times K_2 \times K_3) \cup (\partial D_1 \times \overline{D}_2 \times \overline{D}_3)\}$, *while it is known that it is* 1) *holomorphic in the region* $D_1 \times K_2 \times K_3$ *and continuous at* z_1 *in* $\overline{D}_1$ *for any* $z_k = z_k^0 \subset K_k$ $(k = 2, 3)$; 2) *continuous on the skeleton* $\partial D_1 \times \partial D_2 \times \partial D_3$ *in the set of its variables*; 3) *for any* $z_k^0 \in \partial D_k$ $(k = 1, 2)$ *the function* $f(z_1^0, z_2^0, z_3)$ *is continuous at* z_3 *in* $\overline{D}_3$ *and holomorphic in* D_3; 4) *for any* $z_k^0 \in \partial D_k$ $(k = 1, 3)$ *the function* $f(z_1^0, z_2, z_3^0)$ *is continuous at* z_2 *in* $\overline{D}_2$ *and holomorphic in* D_2.

Then in the region $D_1 \times D_2 \times D_3$ *there exists a holomorphic function* $\tilde{f}(z_1, z_2, z_3)$, *whose values at the points of the region* $D_1 \times K_2 \times K_3$ *coincide with the values of the given function* $f(z_1, z_2, z_3)$.

6. **Bounded functions.** From Hartogs' Theorem 6.2 proved above one may obtain the following theorems dealing with bounded functions.

THEOREM 6.5. *The function* $f(w, z)$ *is holomorphic at every point of the closed bicylinder* $\overline{D}\{|w| \le h,\ |z| \le K\}$ *with the possible exception of the points of the set* $E \subset D$. *In addition:*

1) *For each* $\alpha \in \{|w| \le h\}$ *there exists an at most finite set of points* $P_s(\alpha, \beta_s) \in E$ *while* $|\beta_s| < K_1 < K$ (*here* K_1 *is some constant quantity*).

2) *In each region* D_0, *for* $\overline{D}_0 \subset D$, *the function* $f(w, z)$ *is bounded.*

Then in the closed bicylinder $\overline{D}$ *there exists a holomorphic function* $\tilde{f}(w, z)$ *which coincides at the points of* $D \setminus E$ *with the function* $f(w, z)$.

PROOF. By the hypotheses of the theorem the function $f(w, z)$ is holomorphic in the portion of the region $\overline{D}$ defined by the conditions

$$|w| \le h, \ K_1 \le |z| \le K.$$

Therefore as the region K of the preceding theorem we may take the annulus $K_1 < |z| < K$. Then we consider the function $f(\omega, z)$ of one variable z in the closed disk $|z| \le K$ for $|\omega| = h$. By the hypotheses of the theorem one can find on it an at most finite set of singular points of that function. Because of the boundedness of the function $f(w, z)$ they can be only removable singularities After an appropriate revision or extension of the values of the function $f(w, z)$ at these points we construct a function $\tilde{f}(w, z)$ for which all the assumptions

of Theorem 6.3 are satisfied. Applying that theorem, we thereupon obtain the required result.

Theorem 6.5 holds also for the case of n variables. Here one must require the holomorphicity of the function $f(z_1, \cdots, z_n)$ in the entire closed polycylinder $D\{\,|z_k| \le r_k,\ k = 1, \cdots, n\}$, with the possible exclusion of some set E about which it is known that to each point $(\alpha_1, \cdots, \alpha_{n-1})$ of the closed polycylinder $\{\,|z_k| \le r_k,\ k = 1, \cdots, n-1\}$ there corresponds an at most finite set $(\alpha_1, \cdots, \alpha_{n-1}, \beta_s)$, where $s = 1, 2, \cdots$, lying in E, while always $|\beta_s| < r_n' < r_n$. If, further, the function $f(z_1, \cdots, z_n)$ is bounded in each region D_0 lying along with its frontier in the polycylinder D, then its values at the points of the set E may be so revised or extended that the function turns out to be holomorphic in the entire closed polycylinder $\overline{D}$.

Furthermore it is evident that Theorem 6.5 is true also for a polycylinder with its center at any point of the space C^n.

THEOREM 6.6. *If the function $f(w, z)$ is bounded and holomorphic throughout some neighborhood of the point $P(a, b)$ with the possible exception of the points of the analytic set $\Pi\{g(w, z) = 0\}$ containing the point P, then in some complete neighborhood U_P of P there exists a holomorphic function $\tilde{f}(w, z)$ coinciding at the points of $U_P \setminus \Pi$ with the function $f(w, z)$. Here $g(w, z)$ is some function holomorphic at the point P.*

PROOF. Because of the preparation theorem of Weierstrass the function $g(w, z)$ may be replaced in the neighborhood of the point P (which we place at the origin of coordinates) by a distinguished pseudopolynomial with center at that point. Therefore it follows that there can exist only a finite set of planes $\alpha w + \rho z = 0$ on which the function $g(w, z) = 0$.

One may always assume that the plane $w = 0$ does not belong to that set. In the contrary case we may make an appropriate linear transformation of the space C^n into itself to obtain new variables, which we shall again denote by w and z, for which the above requirement is observed.

Now, applying the preparation theorem of Weierstrass to the function $g(w, z)$, we represent the equation of the set Π in some neighborhood of the point P in the form

$$z^m + \alpha_1(w) z^{m-1} + \cdots + \alpha_m(w) = 0.$$

Suppose that the roots of this equation are $z = g_k(w)$, where $g_k(0) = 0\,(k = 1, \cdots, m)$.

Then, by the hypotheses of our theorem, in some neighborhood of the point P, for example in the bicylinder $U_P\{|w|<h, \ |z|<l\}$, the holomorphicity of the function $f(w, z)$ can be violated only at the points of the surfaces $z = g_k(w)$ $(k = 1, \cdots, m)$. For each positive number $l_1 < l$ there exists a positive number $h_1 = h_1(l_1) < l$ such that for $|w| < h_1$ we will have $|g_k(w)| < l_1$ $(k = 1, \cdots, m)$. Then in the bicylinder $\{|w| < h_1, \ |z| < l\}$ on any analytic surface $w = w_0$ there lie only m points $(w_0, g_k(w_0))$ $(k = 1, \cdots, m)$ at which the holomorphicity of the function $f(w, z)$ is violated, while for all of these points $|g_k(w_0)| < l_1$. Thus for the given function $f(w, z)$ in the bicylinder U_P all the conditions of the preceding theorem are satisfied. Hence our assertion follows.

The theorem just proved holds also for n variables. In this case the set Π is defined by equations of the form $g(z_1, \cdots, z_n) = 0$.

The theorem just obtained becomes in its turn a special case of a general proposition which, in analogy with the corresponding theorem of the theory of functions of one variable, is called the Riemann theorem on the continuation of holomorphic functions.

THEOREM 6.7 (Riemann's theorem on continuation of holomorphic functions). *Suppose that $D \subset C^n$ is some region, $E \subset D$ some sparse set. If the function f is holomorphic at the points of the set $D \setminus E$ and each point $z \in D$ has some neighborhood V_z such that the function f turns out to be bounded on the set $V_z \cap (D \setminus E)$, then in the region D there exists a unique holomorphic function $\tilde{f}$ coinciding on the set $D \setminus E$ with the function f.*

Theorem 6.7 is a consequence of Theorem 6.6 for n variables. From this theorem there also follows an important property of the roots of pseudopolynomials.

THEOREM 6.8. *Suppose that*

$$F(w, z_1, \cdots, z_n) = w^m + A_1 w^{m-1} + \cdots + A_m \tag{1.83}$$

is a continuous distinguished pseudopolynomial with center at the origin of coordinates. Suppose that $Q(a_1, \cdots, a_n)$ is a point of some sufficiently small neighborhood U of the origin of coordinates, not belonging to the discriminant set of this pseudopolynomial (the set of such points of the neighborhood will be denoted by U^*). Then, continuing the root $w_1(z)$ of the pseudopolynomial from the point Q along an appropriately chosen closed path, we may obtain at the point Q the value of any other root of this pseudopolynomial.

PROOF. First of all it is evident, since by hypothesis the discriminant $D(a_1, \cdots, a_n) \neq 0$, that there will exist a neighborhood V of the point Q entirely belonging to U^*. This follows from the continuity of the discriminant $D(z)$. Therefore there will be an infinite set of curves belonging to U^* and passing through the point Q. Continuing the root $w_1(z)$ along these curves, we obtain in U^* some set of functions $w_k(z)$, where $k = 1, \cdots, l$. All of these are roots of the pseudopolynomial (1.83), and therefore $l \leq m$. We consider the function

$$\Phi(w, z) = w^l + p_1(z) w^{l-1} + \cdots + p_n(z) = (w - w_1) \cdots (w - w_l). \tag{1.84}$$

Here the coefficients $p_k(z)$, $k = 1, \cdots, l$, are symmetric functions of the roots w_k, and therefore they are univalent, analytic and accordingly holomorphic functions of the variables $z_1, \cdots, z_n$ at the points of U^*. At the points of the neighborhood U where the discriminant $D(z) = 0$, the functions $p_k(z)$ remain finite. This follows from the fact that the reciprocals of our roots $w' = 1/w$ satisfy the equation $1 + A_1 w' + \cdots + A_n w'^n = 0$ and accordingly may be as small as desired. Hence on the basis of Theorem 6.6 one may conclude that the functions $p_k(z)$ may be analytically continued to the points of the discriminant set which lie in the neighborhood U. Thus the expression (1.84) defines in this neighborhood some pseudopolynomial all the roots of which are roots of the pseudopolynomial $F(w, z)$. Hence it follows that the pseudopolynomial $F(w, z)$ is divided by the pseudopolynomial $\Phi(w, z)$. This last contradicts the assumption that it is irreducible. Accordingly the pseudopolynomials F and Φ must coincide with one another, $l = m$, and our theorem is proved.

We will say that the roots $w_1, \cdots, w_s$ of a pseudopolynomial form a *closed system* in the neighborhood of the center if all of them are obtained from one of them by analytic continuation in this neighborhood. We have proved that all the roots of an irreducible pseudopolynomial form in the neighborhood of the center a closed system.

It is easy to sketch the geometric meaning of the theorem just proved. Consider the space C^{n+1} of variables $w, z_1, \cdots, z_n$. An element of the analytic surface Γ^n in the neighborhood of one of its points P is defined by the equation $f(w, z_1, \cdots, z_n) = 0$. Here f is holomorphic, irreducible at that point. The equation $f = 0$ may be replaced by an equation of type (1.83). The fact that, by the theorem just proved, all of the roots of the pseudopolynomial (1.83) form a closed system has the following significance: let Δ be the collection of

points of the surface Γ^n which project onto the discriminant set of the pseudopolynomial (1.83), defined in the space C^n of the variables $z_1, \cdots, z_n$ by the equation $D(z_1, \cdots, z_n) = 0$. Then *all the points of the surface* Γ^n *in some neighborhood of the point* $P \in \Delta$ *may be connected to one another on* Γ^n, *bypassing* Δ.

7. **Theorem on the equivalence of continuous and analytic continuation.**

THEOREM 6.9. *If it is known that the function* $f(w, z)$ *is* 1) *continuous in some neighborhood* U *of the point* $P(a, b)$ *and* 2) *holomorphic at all the points of the neighborhood* U *which do not belong to some continuously differentiable hypersurface* Σ *containing* P *as one of its ordinary points, then the function is holomorphic in a complete neighborhood of the point* P.

PROOF. We move the origin to the point P. From our hypotheses the equation of the hypersurface Σ may be represented in the neighborhood of the point P as ($w = u + iv$, $z = x + iy$):

$$y = \phi(u, v, x) \tag{1.85}$$

(P is an ordinary point of the hypersurface because the equation of the latter may be expressed in the neighborhood of the point P by means of one of the variables; let that variable be y). Suppose further that the neighborhood U is defined by the inequalities $|w| < k$, $|x| < h$, $|y| < l$. From the continuity of the function $\phi(u, v, x)$ it follows that for $|w| \leq k_1 < k$, $|x| \leq h_1 < h$ we will have $|\phi(u, v, x)| < l_1$, where the numbers k_1, h_1 are chosen so that $l_1 < l$ (recall that $\phi(0, 0, 0) = 0$). Thus all the points of the bicylindrical region

$$U_1\{|w| \leq k_1, \ |x| \leq h_1, \ |y| \leq l_1\},$$

where the function $f(w, z)$ by hypothesis is not assumed holomorphic, lie in the portion of U_1 defined by the inequalities $|w| \leq k_1$, $|x| \leq h$, $|y| \leq l_1$, while in the portion of U_1 defined by the inequalities $|w| \leq k_1$, $|x| \leq h_1$, $l_1 \leq |y| \leq l$, the function $f(w, z)$ is holomorphic. We again apply Theorem 6.2, taking as the region D the disk $|w| < k_1$ and as the region K one of the rectangles defined by the inequalities $|x| < h_1$, $l_1 < |y| < l$. Evidently statement 1) of Theorem 6.2 is satisfied in our case. In order to verify that statement 3) is satisfied, we must consider $f(w, z)$ as a function of z in the rectangle $|x| < h_1$, $l_1 < |y| < l$ for a fixed $w_0 = u_0 + iv_0$ (with $|w_0| = k$). By the hypotheses of our theorem the function $f(w_0, z)$ is holomorphic in this rectangle, with the possible exclusion of points belonging at the same time to the hypersurface Σ. These

points form a smooth curve $y = \phi(u_0, v_0, x)$ in the rectangle. Moreover, it is known that $f(w_0, z)$ is a continuous function on the entire rectangle. Therefore it follows [1] that the function $f(w_0, z)$ is holomorphic in the entire rectangle. Thus statement 3) of Theorem 6.2 is verified. That statement 2) is also satisfied is obvious in view of the hypotheses of the theorem being proved. Our theorem is proved.

The theorem just proved is true in the general case of n variables as well. In this case one need only replace the three-dimensional hypersurface in the theorem with a $(2n-1)$-dimensional hypersurface and the equation (1.85) by the equation

$$y_n = \phi(x_1, y_1, \cdots, x_n). \tag{1.86}$$

§7. HOLOMORPHIC MAPPINGS

1. Basic definitions. Mappings with nonzero Jacobians. Suppose that we are given m functions

$$(T) \qquad w_k = w_k(z_1, \cdots, z_n),\ k = 1, \cdots, m, \tag{1.87}$$

holomorphic on some set $D \subset P^n_z$. They put into correspondence with each point $z \in D$ some point $w \in C^m_w$. The set of points w corresponding to various points of the set D will be denoted by Δ. We shall say that the relations (1.87) define a *holomorphic* mapping of the set $D \subset P^n_z$ onto the set $\Delta \subset C^m_w$ and denote it variously by $z \longrightarrow w$, $w = Tz$, $D \longrightarrow \Delta$, $\Delta = TD$. If $\Delta \subset \mathfrak{D} \subset C^m_w$, where $\mathfrak{D}$ is again some set of points $w \in C^m_w$, then we will say that T is a holomorphic mapping of the set D into the set $\mathfrak{D} \subset C^m_w$ and denote it by $D \longrightarrow \mathfrak{D}$. The mapping $D \longrightarrow \Delta$ is sometimes called a *surjective* holomorphic mapping. If the set $E \subset D$, then relations (1.87) define a holomorphic mapping of the set E into the set Δ. This mapping is said to be a *restriction* of the mapping T onto the set E and is denoted by the symbol $T|_E$.

Consider the case when the sets D and Δ are open. Then from the rules for differentiation the mapping (1.87) will have the following property: If the function $\phi = \phi(w_1, \cdots, w_m)$ is holomorphic in some open set $\Delta_1 \subset \Delta$, the function

1) See for example Courant, *Geometrische Funktionentheorie*, Verlag von Julius Springer, Berlin, 1929.

$$\phi \circ T = \phi(w_1(z), \cdots, w_m(z))$$

will be holomorphic on the open set D_1. Here $\circ$ is the sign of composition, the set D_1 the complete preimage of the open set Δ_1 under the mapping (1.87). This property of the mapping (1.87) evidently includes the property of holomorphicity of the functions $w_k(z)$, since one may always put $\phi(w_1, \cdots, w_m) = w_k$ $(k = 1, \cdots, m)$. This may be taken as the basis of the definition of a holomorphic mapping.

In the sequel we shall usually consider the case when the sets D and Δ are regions.

In the case when $m = n$, we consider a holomorphic mapping of the open set D into the open set Δ

$$(T) \qquad w_k = w_k(z_1, \cdots, z_n), \quad k = 1, \cdots, n. \tag{1.88}$$

If not only the mapping itself but also its inverse mapping

$$(T^{-1}) \qquad z_k = z_k(w_1, \cdots, w_n), \quad k = 1, \cdots, n, \tag{1.89}$$

is holomorphic, then such a mapping is said to be *biholomorphic* or *pseudoconformal.* Evidently a biholomorphic mapping is also a homeomorphism. The inverse is denoted variously by $w \longrightarrow z$, $z = T^{-1}w$, $T(D) \longrightarrow D$, $D = T^{-1}T(D)$.

If the point $z^0 \in P_z^n$, and the point $w^0 = Tz^0 \in C_w^n$, and if there exists a neighborhood of the point z^0 which the mapping T biholomorphically maps onto some neighborhood of the point w^0, then the mapping T is said to be *biholomorphic* or *pseudoconformal* at the point z^0. The term "pseudoconformal" is explained by the fact that for $n > 1$ a biholomorphic mapping generally speaking does not preserve the angles between directions (see in this connection subsection 3 of the following section).

From general theorems of analysis on implicit functions (see subsection 5 of §2 of this chapter) there results the following:

THEOREM 7.1. *If the Jacobian* $\partial w/\partial z = \partial(w_1, \cdots, w_n)/\partial(z_1, \cdots, z_n)$ *of the holomorphic mapping* T, *defined by the equations* (1.88), *is different from zero at the point* $z^0 \in C^n$, *then this mapping is biholomorphic at this point* z^0.

A theorem inverse to Theorem 7.1 is proved in the following subsection of this section.

REMARK 1. We note further that in the case at hand we have the equation

$$\frac{\partial(w_1,\cdots,w_n)}{\partial(z_1,\cdots,z_n)}\cdot\frac{\partial(z_1,\cdots,z_n)}{\partial(w_1,\cdots,w_n)}=1. \tag{1.90}$$

This relation follows from the fact that for holomorphic functions of complex variables the usual rule of differentiation is preserved.

REMARK 2. From the usual rules of differentiation it also follows that for biholomorphic mappings the exterior differential form (1.20_1) is preserved.

2. Algebraic investigation of equations (1.88). This is based on the following theorem of Osgood.[1] We formulate it for the case of n variables and carry out the proof for the case of two variables.

THEOREM 7.2 (Osgood). *Suppose that the functions $f_k(z_1,\cdots,z_n)$ are holomorphic at the point $(a_1,\cdots,a_n)\in C^n$ and $f_k(a_1,\cdots,a_n)=b_k$ (here and in what follows $k=1,\cdots,n$). Then if in some neighborhood of the point $(a_1,\cdots,a_n)$ there are no more points where the equations*[2]

$$f_k(z_1,\cdots,z_n)-b_k=0 \tag{1.91}$$

are satisfied, then the functions $z_k=z_k(w_1,\cdots,w_n)$, satisfying the equations

$$f_k(z_1,\cdots,z_n)=w_k \tag{1.92}$$

and the conditions $z_k(b_1,\cdots,b_n)=a_k$, possess the following properties, possibly after some linear substitution of variables:

1) *The function z_n satisfies the equation*

$$H(z_n,w_1,\cdots,w_n)=(z_n-a_n)^m+$$
$$+\alpha_1(w_1,\cdots,w_n)(z_n-a_n)^{m-1}+\cdots+\alpha_m(w_1,\cdots,w_n)=0, \tag{1.93}$$

where H is a distinguished pseudopolynomial with center at the point $(b_1,\cdots,b_n)$.

2) *The values of the quantities $z_1,\cdots,z_{n-1}$, corresponding to some root z_n, are determined by the formulas*

$$z_q=\frac{g_q(z_n,w_1,\cdots,w_n)}{\Gamma'_{z_n}(z_n,w_1,\cdots,w_n)},\quad q=1,\cdots,n-1. \tag{1.94}$$

1) See Osgood [1], p. 135.

2) The contrary case holds for example if the functions $f_k(z_1,\cdots,z_n)-b_k$ are reducible and have a common factor which vanishes at the point $(z_1=a_1,\cdots,z_n=a_n)$.

Here $\Gamma(z_n, w_1, \cdots, w_n)$ *is the irreducible divisor of the pseudopolynomial* H *to which the given root* z_n *belongs, and* $g_q(z_n, w_1, \cdots, w_n)$ *is a distinguished pseudopolynomial with center at the same point* $(b_1, \cdots, b_n)$. *This pseudopolynomial is the same for all roots* z_n *belonging to the same irreducible divisor of the pseudopolynomial* $H(z_n, w_1, \cdots, w_n)$.

PROOF. First of all we place the origin of coordinates for the spaces z_1, z_2 and w_1, w_2 at the points (a_1, a_2) and (b_1, b_2), respectively. After this we will have $a_1 = a_2 = b_1 = b_2 = 0$. We consider the functions $f_1(z_1, z_2)$, $f_2(z_1, z_2)$. From the preparation theorem of Weierstrass it follows that these functions can vanish identically only on a finite set of analytic planes of the form $Az_1 - Bz_2 = 0$. Therefore there exists an infinite set of such planes at whose points (within some neighborhood of the origin of coordinates) not one of the functions $f_1(z_1, z_2)$, $f_2(z_1, z_2)$ vanishes anywhere except at the origin. Suppose that $\alpha z_1 - \beta z_2 = 0$ is one of these planes.

If we may now choose $\alpha = 0$, $\beta \neq 0$, then this means that $f_1(z_1, 0) \not\equiv 0$, $f_2(z_1, 0) \not\equiv 0$. In the contrary case we make a substitution of variables $z_1 = \beta u_1 + \gamma u_2$, $z_2 = \alpha u_1 + \delta u_2$ (the choice of γ, δ here is subject only to the requirement $\alpha\gamma - \beta\delta \neq 0$). Then

$$f_k(z_1, z_2) = f_k(\beta u_1 + \gamma u_2, \alpha u_1 + \delta u_2) = \phi_k(u_1, u_2)$$

and

$$\phi_k(u_1, 0) = f_k(\beta u_1, \alpha u_1) \not\equiv 0,$$

since from the equations $z_1 = \beta u_1$, $z_2 = \alpha u_1$ it evidently follows that $\alpha z_1 - \beta z_2 = 0$, and in this case, $f_k(z_1, z_2) \not\equiv 0$. Then we return to the original notations. Thus one may always suppose that

$$f_1(z_1, 0) \not\equiv 0, \quad f_2(z_1, 0) \not\equiv 0.$$

Then we apply to the functions $w_1 - f_1(z_1, z_2)$, $w_2 - f_2(z_1, z_2)$ the preparation theorem of Weierstrass and we expand them in powers of z_1. This we may do, since now $f_1(z_1, 0) \not\equiv 0$, $f_2(z_1, 0) \not\equiv 0$. Then the system of equations (1.87) is replaced, since we are considering only those solutions $z_k(w_1, w_2)$ for which $z_k(0, 0) = 0$, by the following:

$$\left.\begin{aligned} H_1 &= z_1^n + \lambda_1(z_2, w_1, w_2)\, z_1^{n-1} + \cdots + \lambda_n(z_2, w_1, w_2) = 0, \\ H_2 &= z_1^m + \mu_1(z_2, w_1, w_2)\, z_1^{m-1} + \cdots + \mu_m(z_2, w_1, w_2) = 0. \end{aligned}\right\} \qquad (1.95)$$

Here H_1, H_2 are distinguished pseudopolynomials with centers at the origin of coordinates. We eliminate z_1, i.e., we set up the resultant of these two equations. This leads us to the equation

$$R(z_2, w_1, w_2) = 0. \tag{1.96}$$

Here $R(z_2, w_1, w_2)$ is a holomorphic function of its variables, since it is formed by addition, subtraction, and multiplication from the coefficients of the equations (1.95). Further, $R(z_2, 0, 0) \not\equiv 0$.

Indeed, if we had $R(z_2, 0, 0) \equiv 0$, this would mean that both of the equations (1.91), where $n = 2$, $b_1 = b_2 = 0$, have at least one common root of the form $z_1 = \phi(z_2)$ (where $\phi(0) = 0$; the possible common roots of these equations for which $\phi(0) \neq 0$ were discarded when we went to the equations (1.95) from the equations (1.92)).[1] But in this case equations (1.91) (considered for the case $n = 2$) would have an infinite set of common solutions in any neighborhood of (0, 0), which contradicts the hypotheses of the theorem. We apply to the function $R(z_2, w_1, w_2)$ the preparation theorem of Weierstrass and find that each function $z_2(w_1, w_2)$ satisfying the system (1.92) and the condition $z_2(0, 0) = 0$ is a root of the equation

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1) For example, if the system (1.92) has the form $w_1 = (1 + z_1 + z_2) z_1$, $w_2 = (1 + z_1 + z_2) z_2$, then system (1.91) would appear as follows: $(1 + z_1 + z_2) z_1 = 0$; $(1 + z_1 + z_2) = 0$. These equations have the common factor $1 + z_1 + z_2$, which however does not vanish at the point (0, 0). Because of the presence of this factor the resultant of these equations is equal to zero. Nevertheless,

$$(1 + z_1 + z_2) z_1 - w_1 =$$
$$= \left[z_1 + \frac{1 + z_2}{2} + \sqrt{\frac{(1 + z_2)^2}{4} + w_1}\right] \left[z_1 + \frac{1 + z_2}{2} - \sqrt{\frac{(1 + z_2)^2}{4} + w_1}\right].$$

The first of the two factors does not vanish at the origin of coordinates, and therefore the pseudopolynomial H_1 is equal to the second factor. The pseudopolynomial H_2 coincides with the left part of the second equation, which is linear in z_1. The resultant $R(z_2, w_1, w_2)$ for these equations already has the properties indicated in the text, i.e., $R(z_2, 0, 0) \not\equiv 0$. The situation would be different if the equation (1.92) had for example the form $w_1 = (z_1 + z_2) z_1$; $w_2 = (z_1 + z_2) z_2$. Here the assumption is not satisfied that there are no solutions of the system (1.91) (other than $z_1 = z_2 = 0$) lying in the neighborhood of the point $z_1 = 0$, $z_2 = 0$, since both equations (1.91) are satisfied by all values of z_1, z_2 for which $z_1 + z_2 = 0$. In this case, $R(z_2, 0, 0) \equiv 0$.

$$H(z_2, w_1, w_2) = z_2^n + \alpha_1(w_1, w_2) z_2^{n-1} + \cdots + \alpha_n(w_1, w_2) = 0. \qquad (1.93')$$

Here $H(z_2, w_1, w_2)$ is a distinguished pseudopolynomial with center at the origin of coordinates. Thus the first of the assertions of the theorem is proved.

Further we factor the pseudopolynomial $H(z_2, w_1, w_2)$ into irreducible factors. Suppose that $\Gamma(z_2, w_1, w_2)$ is one of them, and that $z_2^{(k)}(w_1, w_2)$ is one of the roots of the pseudopolynomial (1.93′) satisfying the equation

$$\Gamma(z_2, w_1, w_2) = z_2^p + \nu_1(w_1, w_2) z_2^{p-1} + \cdots + \nu_p(w_1, w_2) = 0. \qquad (1.97)$$

We substitute this into equation (1.95) and, in order to find the value of z_1 corresponding to this value of z_2, we find the greatest common divisor of the pseudopolynomials obtained from (1.95) after this substitution.

Suppose that $d_k(z_1, z_2^{(k)}, w_1, w_2)$ is this greatest common divisor. We set up the equation

$$h_k(z_1, z_2^{(k)}, w_1, w_2) =$$
$$= z_1^q + \pi_1^{(k)}(z_2^{(k)}, w_1, w_2) z_1^{q-1} + \cdots + \pi_q^{(k)}(z_2^{(k)}, w_1, w_2) = 0. \qquad (1.98)$$

It is obtained from the equation $d_k(z_1, z_2^{(k)}, w_1, w_2) = 0$ as the result of replacement of any multiple roots which might occur in it by simple ones. Thus it has the same roots as the equation $d_k(z_1, z_2^{(k)}, w_1, w_2) = 0$ and gives the value of the unknown z_1, corresponding to the value $z_2 = z_2^{(k)}(w_1, w_2)$. On continuing the function $z_2^{(k)}(w_1, w_2)$ from some point Q along every possible closed path lying in some neighborhood of the origin and having no common points with the discriminant set of the equation $\Gamma(z_2, w_1, w_2) = 0$, we obtain at Q the values of all the other roots of the equation (1.97). This follows from Theorem 6.8. Under this continuation the equation (1.98) will at the same time go into an equation for the root z_1 corresponding to the other roots $z_2^{(k)}$. Thus it follows in particular that q (which must vary continuously and yet remain an integer) must have one and the same value for all k.

We form also the equation

$$F(z_1, w_1, w_2) = \prod_{k=1}^{p} h_k(z_1, z_2^{(k)}, w_1, w_2) = 0. \qquad (1.99)$$

The function $F(z_1, w_1, w_2)$ is univalent and analytic and accordingly a holomorphic function of the variables w_1, w_2 in the neighborhood of the origin, since

on passing from one root $z_2^{(k)}$ to another, the factors of (1.98) are only rearranged by the continuation and the entire product does not change. Further, it is evident that $F(z_1, 0, 0) \not\equiv 0$. Thus, all the assumptions of the preparation theorem of Weierstrass are satisfied, and from this theorem we may substitute for equation (1.99) (since we are interested only in functions $z_1(w_1, w_2)$ for which $z_1(0, 0) = 0$) the equation

$$P(z_1, w_1, w_2) = z_1^r + \rho_1(w_1, w_2) z_1^{r-1} + \cdots + \rho_r(w_1, w_2) = 0. \qquad (1.100)$$

Here $P(z_1, w_1, w_2)$ is a distinguished pseudopolynomial with center at the origin of coordinates. As we have indicated, to the value $z_2^{(k)}$ there correspond q values of z_1. We replace z_2 in the equations (1.95) by a new variable $z_2 + \eta z_1$. We choose the quantity η so that all the $z_2^{(k)} + \eta z_1^{(s)}$ (here $z_2^{(k)}$ is any root of the equation (1.93′) and $z_1^{(s)}$ is any root of the equation $F = 0$ or of one analogous to it obtained by choosing instead of Γ another irreducible factor of the pseudopolynomial (1.93′)) will be distinct at some point Q of a small neighborhood of the origin. To this end, in the assignment of η we must avoid only a certain finite set of values defined by the equations $z_2^{(k_1)} + \eta z_1^{(s_1)} = z_2^{(k_2)} + z_1^{(s_2)}$. Putting $z_1 = v_1$, $z_2 = v_2 - \eta v_1$ in equations (1.95), we obtain instead of equations (1.97) and $F = 0$ a new system, while to each root v_2 there will correspond only one root v_1. Note that both substitutions used in our discussions, the transition from the variables z_1, z_2 to the variables u_1, u_2 and then to the variables v_1, v_2, may be unified into one substitution. So the equation (1.98) either immediately turns out to be linear in z_1 or else will reduce to a linear equation. After carrying out the indicated substitution, we then return again to the initial notations for the variables and may suppose that the above system of equations for z_1, z_2 already has the required property: to each root z_2 of the equation (1.93′) there corresponds only one root z_1. In this case, evidently $q = 1$ and $r = p$.

It remains for us only to show that the value z_1 corresponding to some root z_2 may be represented in the form (1.94). To this end we consider the expressions

$$\left.\begin{aligned} Q_0 &= z_1^{(1)} + z_1^{(2)} + \cdots + z_1^{(p)}, \\ Q_1 &= z_2^{(1)} z_1^{(1)} + \cdots + z_2^{(p)} z_1^{(p)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Q_{p-1} &= (z_2^{(1)})^{p-1} z_1^{(1)} + \cdots + (z_2^{(p)})^{p-1} z_1^{(p)}. \end{aligned}\right\} \qquad (1.101)$$

The quantities Q_k are symmetric functions of the roots of the pseudopolynomials (1.97) and (1.100). Therefore they are univalent and analytic and accordingly holomorphic at the points of some neighborhood of the point (0, 0) not belonging to the discriminant sets of these pseudopolynomials. They are finite (as are the functions $z_1^{(s)}$, $z_2^{(t)}$ from which they are made up) in this entire neighborhood. By Theorem 6.6 they will be holomorphic functions of the variables w_1, w_2 in the whole of this neighborhood.

Further it is evident that

$$\left.\begin{aligned} \Gamma(z_2, w_1, w_2) &= (z_2 - z_2^{(1)})\cdots(z_2 - z_2^{(p)}), \\ \Gamma'_{z_2}(z_2^{(1)}, w_1, w_2) &= (z_2^{(1)} - z_1^{(2)})\cdots(z_2^{(1)} - z_2^{(p)}). \end{aligned}\right\} \tag{1.102}$$

Put

$$L(z_2, w_1, w_2) = (z_2 - z_2^{(2)})\cdots(z_2 - z_2^{(p)}) = \\ = z_2^{p-1} + c_1 z_2^{p-2} + \cdots + c_{p-1}. \tag{1.103}$$

Obviously

$$c_1 = \nu_1 + z_2^{(1)}; \quad c_2 = \nu_2 + z_2^{(1)}\nu_1 + (z_2^{(1)})^2, \tag{1.104}$$

where the ν_k are the coefficients of equation (1.97). The c_{n-k} turn out to be polynomials of degree $p - k$ with respect to $z_2^{(1)}$. Multiplying equations (1.101) respectively by $c_{p-1}, c_{p-2}, \cdots, c_0 = 1$ and then adding them and using the fact that for $k = 2, 3, \cdots, p$, the expression $L(z_2^{(k)}, w_1, w_2) = 0$, and $L(z_2^{(1)}, w_1, w_2) = \Gamma'_{z_2}(z_2^{(1)}, w_1, w_2)$, we obtain

$$z_1^{(1)}\Gamma'_{z_2}(z_2^{(1)}, w_1, w_2) = g(z_2^{(1)}, w_1, w_2). \tag{1.105}$$

Here $g(z_2^{(1)}, w_1, w_2)$ is a pseudopolynomial expanded in powers of $z_2^{(1)}$, as follows from the holomorphicity of the functions $Q_k(w_1, w_2)$ and equations (1.104).

By continuation along different paths passing through some point R of a neighborhood of the origin of coordinates we discover that the relation (1.105) holds also for all remaining pairs of values of the variables z_1, z_2. Thus it has been established that the root z_1 corresponding to some root z_2 may be expressed in terms of the equation (1.94):

$$z_1 = \frac{g(z_2, w_1, w_2)}{\Gamma'_{z_2}(z_2, w_1, w_2)}.$$

Our theorem is completely proved.

REMARK. From the finiteness of z_1 it obviously follows that the pseudopolynomial $g(z_2, w_1, w_2)$ vanishes at the points of the discriminant set of the pseudopolynomial $\Gamma(z_2, w_1, w_2)$.

We note some consequences of Osgood's theorem, connected with the mapping defined by equation (1.88).

If at the point $a(a_1, \cdots, a_n) \in C^n$ the Jacobian $\partial w/\partial z \neq 0$, then as we know from Theorem 7.1, equations (1.88) define a biholomorphic mapping of some neighborhood U_a of that point a onto some neighborhood V_b of the point $b(b_1, \cdots, b_n) \in C^n$, where $b_k = w_k(a_1, \cdots, a_n)$, $k = 1, \cdots, n$. In this case the hypotheses of Osgood's theorem on the system of equations (1.91) are satisfied, and accordingly its conclusion also holds. In addition the degree of the pseudopolynomial (1.93) there is equal to unity, since in the contrary case the mapping T defined by equations (1.88) cannot be one-to-one.

If at the point $a \in C^n$ the Jacobian $\partial w/\partial z = 0$, then two cases are possible: 1) The hypotheses of Osgood's theorem on the system of equations (1.91) are not satisfied, in which case the mapping T is trivially not one-to-one. 2) The hypotheses of Osgood's theorem on the system of equations (1.91) are satisfied. We shall show that in this case the degree of the pseudopolynomial (1.93) always will be larger than unity and accordingly the mapping T in some neighborhood of the point a is also not one-to-one.

Indeed, if this degree is equal to unity, then from equations (1.93) and (1.94) there is defined a system of functions $z_k(w_1, \cdots, w_n)$ $(k = 1, \cdots, n)$ that are holomorphic at the point b. In some neighborhood of this point the Jacobian $\partial z/\partial w$ will be a holomorphic function of its variables, and in addition we will have there the identity

$$w_k(z_1(w), \cdots, z_n(w)) \equiv w_k, \quad k = 1, \cdots, n. \tag{1.106}$$

From equation (1.106) by the usual rules one obtains identity (1.90). However, this is impossible, since by hypothesis the Jacobian $(\partial w/\partial z)_a = 0$, and the Jacobian $\partial z/\partial w$ is holomorphic at the point b. Thus we arrive at the following important theorem.

THEOREM 7.3. *If the holomorphic mapping* T *(defined by equations* (1.88)) *is biholomorphic at the point* $z = a \in C^n$, *then its Jacobian* $\partial w/\partial z$ *is different from zero at this point.*

In other words: *if the Jacobian* $(\partial w/\partial z)_a = 0$, *then in every neighborhood of the point* a *there are distinct points which the mapping* T *carries into one point.*

Holomorphic mappings with a Jacobian equal to zero will be considered in more detail in the following chapter (see subsection 2 of §10).

3. Distortion for pseudoholomorphic mappings. We restrict ourselves to the case of two variables w, z and consider the pseudoconformal mapping

$$W = W(w, z), \quad Z = Z(w, z) \tag{1.107}$$

of some neighborhood of the point $P(w_0, z_0)$ onto some neighborhood of the point $P^*(W_0, Z_0)$, where $W_0 = W(w_0, z_0)$, $Z_0 = Z(w_0, z_0)$.

Under the mapping (1.107) the angles of rotation, in the same way as the coefficients of linear distortion for lines passing through the point P in *distinct* directions, turn out in general to be distinct.

This circumstance defines the name itself of the mapping, which is referred to as "pseudoconformal" in distinction from a conformal mapping $Z = f(z)$, where $f(z)$ is a holomorphic function of one complex variable z. Our object is to indicate how the lengths and directions change under the mapping (1.107) and in what way the argument and modulus of the Jacobian $(\partial w/\partial z)_P = J(w_0, z_0) = J$ are connected with these changes.

Evidently it is sufficient for this purpose to consider the linear mapping

$$W = aw + bz, \quad Z = cw + dz, \tag{1.108}$$

which is called the *differential* of the mapping (1.107). Here $a = W'_w(w_0, z_0)$, $b = W'_z(w_0, z_0)$, $c = Z'_w(w_0, z_0)$, $d = Z'_z(w_0, z_0)$. We suppose further $w_0 = z_0 = W_0 = Z_0 = 0$, which does not affect the generality of our deductions. We shall compare the directions and lengths of vectors with complex components w, z and W, Z.

The difference of the lengths of the vector $\{W, Z\}$ from that of the vector $\{w, z\}$ is characterized by the coefficient of linear distortion κ, where $\kappa^2 = (|W|^2 + |Z|^2)/(|w|^2 + |z|^2)$. The deviation of the vector $\{W, Z\}$ from the vector $\{w, z\}$ will be expressed by means of the analytic angles θ and ϕ between these vectors.

Suppose further that $\omega = w/z$, $\Omega = W/Z$ (for $z = 0$ we take $\omega = 0$, and for $Z = 0$ we take $\Omega = \infty$) are the parameters of the one-complex-dimensional analytic planes containing respectively the vectors $\{w, z\}$ and $\{W, Z\}$ (see subsection 8 of §4; there our variable w corresponds to the variable z_2, and the variable z corresponds to the variable z_1). Then from equation (1.108) there follows

$$\Omega = \frac{a\omega + b}{c\omega + d}. \tag{1.109}$$

Further it is easy to see that

$$\kappa^2 = \frac{|a\omega + b|^2 + |c\omega + d|^2}{|\omega|^2 + 1}$$

$$(\text{for } \omega = \infty,\ \kappa^2 = |a|^2 + |c|^2), \tag{1.110}$$

$$\cos\theta\, e^{i\phi} = \frac{(a\omega + b)\bar{\omega} + (c\omega + d)}{\sqrt{|a\omega + b|^2 + |c\omega + d|^2}\sqrt{|\omega|^2 + 1}}$$

$$\left\{\text{for } \omega = \infty,\ \cos\theta\, e^{i\phi} = \frac{a}{\sqrt{|a|^2 + |c|^2}}\right\}, \tag{1.111}$$

$$\sin\theta = \frac{|(a\omega + b) - (c\omega + d)\omega|}{\sqrt{|a\omega + b|^2 + |c\omega + d|^2}\sqrt{|\omega|^2 + 1}} =$$

$$= \frac{|\Omega - \omega|}{\sqrt{(1 + |\omega|^2)(1 + |\Omega|^2)}}$$

$$\left\{\text{for } \omega = \infty,\ \sin\theta = \frac{|c|}{\sqrt{|a|^2 + |c|^2}} = \frac{1}{\sqrt{1 + |\Omega|^2}}\right\}. \tag{1.112}$$

From formulas (1.110) and (1.111) it is evident that for all vectors $\{w, z\}$ contained in the same plane ω (i.e., plane with parameter ω) the quantities κ, θ and ϕ have one and the same value. Therefore under the mapping (1.108): 1) each analytic plane ω goes over into some other analytic plane Ω, forming with the original plane the angle θ; 2) all vectors lying in the plane Ω and issuing from the point P, are then rotated through the angle ϕ; 3) the lengths of all of these vectors are expanded by κ (if $\kappa \geq 1$) or contracted $1/\kappa$ times (if $\kappa \leq 1$). When the angle of the first rotation $\theta = \pi/2$, then the angle of the second rotation ϕ (as results from formula (1.111))

remains undetermined.

Consider the Riemann sphere of the complex variable ω. Then, as we have seen in formula (1.68), $\sin\theta$ is equal to half the length of the chord of that sphere joining its points ω and Ω. For $\theta = \pi/2$ the corresponding points ω and Ω are diametrically opposite to each other. Then $\Omega\overline{\omega} + 1 = 0$ or in other words $a\omega\overline{\omega} + b\overline{\omega} + c\omega + d = 0$ (for the plane $\omega = \infty$ the angle $\theta = \pi/2$, if $a = 0$). This equation leads to the following theorem.

THEOREM 7.4. *Suppose that*

$$\Delta = \frac{b\overline{b} + c\overline{c} - (a\overline{d} + \overline{a}d)}{2\,|\,ad - bc\,|}. \tag{1.113}$$

Then for $\Delta \geq 1$ there exist planes ω for which the angle of the first rotation is equal to $\theta = \pi/2$. For $-1 \leq \Delta < 1$, for all planes ω, the angle of the first rotation is $\theta < \pi/2$. The case $\Delta < -1$ is impossible.

We shall not dwell here on the proof of this theorem. Like the proof of other theorems of the present subsection, it may be found in the appropriate article.[1)]

We shall seek the largest value of the angle of the first rotation θ and the plane ω on which it occurs. Evidently to that plane corresponds the point (sphere) ω which undergoes the greatest displacement under the mapping (1.109). If $\Delta \geq 1$, then the greatest displacement is undergone by the points ω that have diametrically opposite images under the mapping (1.109). For the case $-1 \leq \Delta < 1$ the following theorem is valid:

THEOREM 7.5. *If $-1 \leq \Delta < 1$, then under the mapping (1.108) the angles of the first rotation θ of the planes ω satisfy the inequality*

$$\operatorname{tg}\frac{\theta}{2} \leq \left|\frac{1-\sqrt{k}}{1+\sqrt{k}}\right| \operatorname{ctg}\frac{\sigma}{2} \quad \text{for } \sigma \neq 0, \tag{1.114}$$

$$\operatorname{tg}\frac{\theta}{2} < \frac{|\,l\,|}{2} \qquad\qquad \text{for } \sigma = 0. \tag{1.115}$$

Here σ is the angle between planes remaining invariant under the mapping (1.108), k is the dilation factor of the bilinear mapping (1.109), $\sqrt{k} = |\sqrt{k}|\, e^{(i/2)\arg k}$, and l is the parameter of the mapping (1.109), if it is parabolic.

1) See Fuks [3, 4].

REMARK 1. Inequalities (1.114) and (1.115) are sharp. For the extremal plane the angle of the first rotation $\theta = \theta_0$, while $\operatorname{tg}(\theta_0/2) = |(1-\sqrt{k})/(1+\sqrt{k})| \operatorname{ctg}(\sigma/2)$, or $\operatorname{tg}(\theta_0/2) = |l|/2$; the angle of the second rotation $\phi_0 = \arg J/2$; the coefficient of linear distortion $\kappa_0 = \sqrt{|J|}$. Here $J = ad - bc$ is the Jacobian of the mapping (1.108), or what is the same thing, the Jacobian of the mapping (1.107) at the point P.

REMARK 2. If $\Delta \geq 1$, then the properties listed in the preceding remark are valid for the mapping (1.108) of the plane $\omega = \omega_0$ corresponding to the stationary value of the angle of the first rotation θ_0. For this plane the formal derivative $(\partial/\partial\omega)\operatorname{tg}^2\theta = (\partial/\partial\omega)\,|(\Omega-\omega)/(1+\Omega\bar{\omega})|^2 = 0$ (where $\Omega = (a\omega+b)/(c\omega+d)$). However, in distinction from the case $-1 \leq \Delta < 1$, this stationary value θ_0 of the angle of the first rotation is not extremal. In this case what is extremal is the value of the angle of the first rotation $\theta = \pi/2$.

Now we shall consider the limits for the variation of the angle of the second rotation ϕ. It turns out that the existence of such limits also depends on the values of the quantity Δ. The following theorem holds:

THEOREM 7.6. *If $\Delta > 1$, the angle of the second rotation ϕ for various planes ω takes on all values in the interval $-\pi < \phi \leq \pi$. If $-1 < \Delta < 1$, then the value of the second rotation ϕ for any plane ω is included between the extremal values $\phi^{(1)}$ and $\phi^{(2)}$. These values satisfy the condition*

$$\phi^{(1)} + \phi^{(2)} \equiv \arg J \pmod{2\pi}. \tag{1.116}$$

Here $J = ad - bc$ is the Jacobian of the mapping (1.108).

REMARK. We note that equation (1.116) also holds for angles of the second rotation corresponding to planes ω_1 and ω_2 remaining invariant under the mapping (1.108). If the fractional linear mapping (1.109) is parabolic and $\omega_1 = \omega_2$, then $\phi_1 = \phi_2 = \frac{1}{2}\arg J$.

The relation indicated in our remark makes possible a geometric interpretation of the argument of the Jacobian of the pseudoconformal mapping (1.107) which is valid for any value of Δ. Theorem 7.6 gives such an interpretation only for $-1 < \Delta < 1$.

Theorems 7.4–7.6 show that, depending on the value of the quantity Δ, pseudoconformal mappings differ essentially in their properties. Pseudoconformal mappings with $|\Delta| < 1$ will be called mappings of *bounded scope.* If $\Delta > 1$ the

mappings are said to be of *arbitrary scope.*

In conclusion we observe that the coefficients of linear distortion corresponding to different planes ω are always included between two extremal values κ_1 and κ_2. These latter satisfy the condition $\kappa_1 \kappa_2 = |J|^2$. This follows from the fact that the coefficient of volume distortion of the mapping (1.107) is equal to $|J|^2$. The relation $\kappa_1 \kappa_2 = |J|^2$ also holds for coefficients of linear distortion corresponding to planes ω_1 and ω_2 which remain invariant under the mapping (1.108).

CHAPTER II

FUNDAMENTAL PROPERTIES OF HOLOMORPHIC FUNCTIONS IN PLANE COVERING REGIONS. SINGULAR POINTS

In the study of complete analytic functions of several complex variables and the regions where they are defined as unions of holomorphic functional elements, it is convenient to consider particular geometrical forms analogous to Riemann surfaces. These forms have the name Riemann regions. In general such regions are defined as Hausdorff topological spaces, each of which has a holomorphic mapping onto some region of the space C^n (or complex projective space P^n or the function-theoretic space G^n). However, in many parts of the theory of functions of several complex variables one may restrict oneself, as in the case of one variable, to the consideration of Riemann regions of a special kind, namely, plane covering regions over the space P^n or the space G^n. They are made up in a definite way from regions of these spaces, in the case of one variable from regions of the extended complex plane.

The present chapter is devoted to the study of plane covering regions. In the following chapter we consider Riemann regions of general kind.

§8. PLANE COVERING REGIONS OVER THE SPACE P^n

1. Fundamental definitions. We consider the complex projective space P^n, containing as a subset the space C^n of variables $z_1, \cdots, z_n$. We shall call a point of the space P^n a geometrical point and denote it by underlining. In this space we choose a finite or countable sequence of regions S_i', $i = 0, 1, 2, \cdots$, of one of the following two kinds:

I) The sequence consists of the regions $S_i = \{\chi(\underline{M}, \underline{M}_i) < R_i\}$. Here $\chi(\underline{M}, \underline{M}_i)$ is the chordal distance of the point $\underline{M}$ from the fixed point $\underline{M}_i$, which is called the *center* of the region S_i. The number $R_i > 0$ is called the *radius*

of the region S_i. Usually we write $S_i = S(\underline{M}_i, R_i)$.

II) The sequence consists of regions of two types. The regions S_i of the first type are polycylinders $\{|z_k - z_k^{(i)}| < R_k^{(i)},\ k = 1, \cdots, n\}$; The point $(z_1^{(i)}, \cdots, z_n^{(i)})$ is called the center of the region S_i. One supposes that all the numbers $R_k^{(i)} > 0$ $(k = 1, \cdots, n)$. The regions S_i of the second type are defined by conditions of the type (1.78).

$$\left|\frac{\zeta_1}{\zeta_\nu}\right| < R_1^{(i)}, \cdots, \left|\frac{\zeta_{\nu-1}}{\zeta_\nu}\right| < R_{\nu-1}^{(i)}, \left|\frac{\zeta_{n+1}}{\zeta_\nu}\right| < R_\nu^{(i)},$$

$$\left|\frac{\zeta_{\nu+1}}{\zeta_\nu} - \alpha_{\nu+1}^{(i)}\right| < R_{\nu+1}^{(i)}, \cdots, \left|\frac{\zeta_n}{\zeta_\nu} - \alpha_n^{(i)}\right| < R_n^{(i)}.$$

Here $\zeta_1, \cdots, \zeta_{n+1}$ are homogeneous coordinates in the space P^n, the point $\underline{M}_i(\underbrace{0, \cdots, 0}_{\nu-1 \text{ times}}, \infty, \alpha_{\nu+1}^{(i)}, \cdots, \alpha_n^{(i)})$ is the center of the region S_i. Note that the sequence of regions chosen may consist either of regions S_i of both types of the second kind, or else of regions of one type.

In the particular case when $R_1^{(i)} = \cdots = R_n^{(i)} = R_i$, the number $R_i > 0$ is called the *radius* of the region S_i. Then we write $S_i = S(\underline{M}_i, R_i)$. We note that in what follows it is the regions $S(\underline{M}_i, R_i)$ that we shall consider. General regions S_i of the second kind will be used only in forming the products of covering regions.

An analogous sequence of regions S_i may also be formed for the function-theoretical space G^n. In this case instead of the regions S_i of the second type we must take the images of polycylinders under the corresponding mappings $z_k^* = (a_k z_k + b_k)/(c_k z_k + d_k)$ (where $k = 1, \cdots, n$, $a_k d_k - b_k c_k \neq 0$). Note that a similar method of definition of the regions S_i of the second type may also be applied in the case of the space P^n. However, the definition given above turns out to be more convenient.

In what follows, unless we specify otherwise, we consider regions S_i constructed for the complex projective space P^n. The regions S_i are said to be plane covering regions or plane covers or elements. Here the word "plane" will be omitted where this omission will not lead to misunderstanding.

Further, to each pair of covers S_i and S_j of our sequence we put into correspondence the numbers $\epsilon_{ij} = \epsilon_{ji}$. The choice of the quantities ϵ_{ij} is

subject to the following conditions:

1. If the covers S_i and S_j have no common geometrical points, then $\epsilon_{ij} = 0$.

2. All $\epsilon_{ii} = 1$.

3. If the covers S_i and S_j have common geometrical points, ϵ_{ij} may be taken to be equal to 0 or to 1. The sole restriction laid on the choice between these values is as follows: if there exist geometrical points lying at the same time in the covers S_i, S_j, and S_k, then for $\epsilon_{ij} = 1$ one necessarily has $\epsilon_{ik} = \epsilon_{jk}$.

The set of plane covers S_i and the numbers ϵ_{ij} will be denoted by $\{S_i, \epsilon_{ij}\}$ and called a system of plane covers. The collection consisting of a geometrical point $\underline{M}$ and one of the covers S_i to which it belongs will be called an *analytic point* M. We shall say that the geometric point $\underline{M}$ is *fundamental* with respect to the analytic point M, or, for brevity, that it is its *projection* and that the point M lies above the point $\underline{M}$.

The analytic point M is said to be finite or a point at infinity according to whether it lies above a finite or point at infinity $\underline{M}$.

In the cases when it does not lead to any misunderstanding, geometrical and analytic points will simply be called points, and we shall denote them by letters without underlining, and we shall analogously deal with various sets consisting of these points.

Two analytic points M' and M'' of the system of covers $\{S_i, \epsilon_{ij}\}$ will be considered to be *identical* if their projections coincide, and if for the covers S_i, S_j belonging to them respectively, the coefficient $\epsilon_{ij} = 1$.

DEFINITION (plane covering region). A set D of points of the system of covers $\{S_i, \epsilon_{ij}\}$ is said to be a plane covering region or a plane region over the space P^n (if there is no possibility of causing a misunderstanding, the word "plane" will be omitted) if the covers S_i can be indexed in such a way that for each S_i for $i > 0$ there is in the system $\{S_j\}$ a cover S_j for which $j < i$ and $\epsilon_{ij} = 1$.

The elements S_i form in such a case a *covering* of the covering region D over the space P^n. We agree always (unless the contrary is specified) to enumerate the component covers S_i of the covering in the order indicated in our definition. Evidently if D is some covering region over the space P^n, then the set of its fundamental points also forms a region $\underline{D}$ in the space of the variables $z_1, \cdots, z_n$. The region $\underline{D}$ is called a *fundamental region* for D.

A covering region over the space P^n is said to be *infinite* if it contains analytic points at infinity. A region containing no such points is said to be finite. If all the fundamental points of the region D have coordinates satisfying the condition $|z_1|^2 + \cdots + |z_n|^2 < R^2$ (where R is some positive number), then the region D is said to be *bounded*.

DEFINITION (finite-sheeted covering region). A covering region D over the space P^n is said to be *p-sheeted* over the geometrical point $\underline{M}$ if there are p different analytic points belonging to it for which the point $\underline{M}$ is the projection. If there exists a number p_0 which is the maximum of the numbers p for all the points $\underline{M}$ of the fundamental region $\underline{D}$, then the region D is said to be *finite-sheeted* (p_0-sheeted) over the region $\underline{D}$; if $p_0 = 1$, it is *single-sheeted*.

If there is no such number p_0, then the plane covering region over the space P^n is said to be *infinitely-sheeted*.

Now we suppose that we are given a region D over the space C_z^n of variables $z_1, \cdots, z_n$ defined by the system of covers $\{S_i, \epsilon'_{ij}\}$, and a region E over the space C_w^m of variables $w_1, \cdots, w_m$ defined by the system of covers $\{T_p, \epsilon''_{pq}\}$. We define the *product of the regions D and E*, namely, the region $D \times E$, over the space $C_z^n \times C_w^m$ of variables z, w in the following way, with the aid of the system of covers $\{\Sigma_\alpha, \epsilon_{\alpha\beta}\}$. The cover Σ_α we define by the equation $\Sigma_\alpha = \Sigma_{(ip)} = S_i \times T_j$. The number $\epsilon_{\alpha\beta}$ is taken equal to zero if the covers Σ_α and Σ_β have no common geometrical points. In the contrary case $\epsilon_{(ip),(jq)} = \epsilon_{ij}\epsilon_{pq}$. In this construction we use covers of the second kind.

If to each analytic point $M \in D$ we put into correspondence one or several complex numbers, then we will say that in the region D there is defined a single- or many-valued function $f(M)$. If between the analytic points of two covering regions a relation is established in some way, then we will say that there is given a *mapping* T of one region D_1 onto the other, D_2, denoted by $D_1 \to D_2$, $D_2 = TD_1$.

2. Relations between plane covering regions over the space P^n.

DEFINITION (limit of a sequence). A sequence of analytic points M_m *converges* to the analytic point M, which is the limit of that sequence (in this case we shall write $\lim_{m\to\infty} M_m = M$), if 1) $\lim_{m\to\infty} \underline{M}_m = \underline{M}$, 2) beginning with some $m = N$, $\epsilon_{\beta_m\alpha} = 1$, where S_α is the cover belonging to the analytic point M, and $S_{\beta m}$ is the cover belonging to the analytic point M_m.

This concept of a converging sequence of analytic points will be used in

further definitions of the present part of this paper.

First of all: two regions D_1 and D_2 over the space P^n are said to be *identical* if between the points of these regions one may establish a one-to-one relation with the following properties:

1) Two points corresponding to each other have the same projection.

2) If some sequence of points M_m of one region converges to a point M of this region, then the sequence of corresponding points P_m of the other region converges to the point P which corresponds to the point M.

In this case we shall write $D_1 = D_2$.

We note that as a result of our definitions a single-sheeted region may be considered as identical with its fundamental region. We shall denote them by the same letter.

Further, we shall say that a region or open set D_1 (the latter generally speaking is a collection of covering regions not connected to one another) lies *inside* the region D if we can set up a one-to-one correspondence between the points of D_1 and some subset of the points of D with the following properties:

1) Two points corresponding to one another have the same projection.

2) If some sequence of points M_l' of the region D_1 converges to a point M' of that region, then the sequence of corresponding points M_l of the region D converges to the point M of the region D which corresponds to the point M'.[1)]

In this case we will write $D_1 < D$.

From our definition it follows that the region D_1 may be more ramified than D and still project inside the region D.

Evidently, if $D_1 < D_2$, $D_2 < D_1$, then $D_1 = D_2$.

If the relation between the points of the region D_1 and the part of the points of the region D dealt with in our definitions is one-to-one, then one says that the region D_1 is a *subregion* of the region D, or in other words that *the region D contains the region D_1 as a subregion.*

If we identify the corresponding points of the subregion D_1 and the region D, then we may, as usual, regard this subregion D_1 as a connected open subset of the region D.

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1) We note that in each of the two last definitions the second condition expresses the continuity of the relation (see the following subsection of the present section).

Finally, we shall say that the region D_1 *lies strictly inside the region* D if 1) $D_1 < D$ and 2) if M_l is some sequence of points of the region D_1, then from the sequence of corresponding points $\underline{M}_l$ of the region D one may always single out a subsequence converging to some point M of the region D. (This means that to the boundary points of D_1 there correspond interior points of the region D. For the definition of boundary points of a region see below in subsection 6.) In this case we shall write $D_1 \ll D$.[1)]

In an analogous sense further on we shall require the expression "the subset Q of points of the region D lies *strictly* inside that region," and we shall use the notation $Q \ll D$.

3. Neighborhoods in covering regions over the space P^n.

DEFINITION (neighborhood of an analytic point). Any single-sheeted region containing the point P which is a subregion of D is said to be a neighborhood of the point P in the region D. Such neighborhoods will be denoted by the symbol $V_D(P)$.

Among the various neighborhoods of the point P in the region D a particularly important role is played by the so-called "*elementary neighborhoods* of the point P in the region D." An elementary neighborhood of the point P in the region D will be denoted by $S_D(P)$ and defined as follows: 1) the region $S_D(P)$ is a neighborhood of the point P in the region D; 2) its projection $\underline{S_D(P)}$ is given: a) if one uses elements of the first kind, by the condition: the point $\underline{z} \in \underline{S_D(P)}$, if $\chi(\underline{z}, \underline{P}) < R$, where the number $R > 0$ is taken to be maximal; b) if one is using elements of the second kind for a finite point P, by the condition: the point $\underline{z} \in \underline{S_D(P)}$, if $| z_k - z_k^0 | < R$, $k = 1, \cdots, n$, where the number $R > 0$ is taken as large as possible, $z_1^0, \cdots, z_n^0$ are the coordinates of the point P, $z_1, \cdots, z_n$ are the coordinates of the point z. If P is the point at infinity, then as the projection $\underline{S_D(P)}$ one chooses a region given by the inequalities of type (1.78), where the number $R > 0$ is again taken to be maximal.

The number $R = r_D(P)$ defined in this way is called the *limiting distance* of the point P in the region D. We note that the case $R = \infty$, if one is using

1) If not only $D_1 < D$ but $D_1 \subset D$ (so that D_1 is a subregion of the region D), then one usually writes $D_1 \Subset D$.

covers of the second kind, is not excluded from consideration.

If the set D_1 lies inside the region D and the point $M_1 \in D_1$, then by the limiting distance of the point M_1 in the region D we will understand the limiting distance in the region D of the point M of this region corresponding to the point M_1. If the set D_1 lies inside the region D and the limiting distances of its points M (its finite points if one is using covers of the second kind) in the region D have a lower bound ρ which is positive, then this number ρ is called the *minimal limiting distance* of the set D_1 in the set D. It is clear that such a number ρ always exists and is not equal to zero if the set D_1 lies strictly inside the region D.

Note that one cannot derive the converse conclusion. From the fact that $D_1 < D$ and there exists a minimal limiting distance ρ of the set D_1 in the set D, it does not follow[1)] in general that $D_1 \ll D$. Such a conclusion may be made only for finitely-sheeted regions.

The definition given above for a neighborhood of a point P in a covering region D over the space P^n makes it possible to consider for such points "arbitrarily small" neighborhoods, in fact: let V be some neighborhood of the point M, $\underline{V}$ its projection, $\underline{V_0}$ some region contained inside the region $\underline{V}$ and including the point $\underline{M}$. Then for any region $\underline{V_0}$ (for example, it may be a polycylinder of radius ϵ with center at the point $\underline{M}$, where ϵ is an arbitrarily small number), one can always find a neighborhood $U_D(M)$ such that $\underline{U_D}(M)$ is contained in the region $\underline{V_0}$.

The construction of neighborhoods of analytic points makes it possible to formulate in the usual way the concepts of limit of a sequence of analytic points, continuous function in a covering region over the space P^n, and continuous mapping of one such region onto another. In this connection, as usual, we consider the concept of continuity only as applied to univalent functions and to mappings which are single-valued. It is also useful to observe in particular that this concept refers both to finite regions and to regions at infinity over the space P^n. In particular, the image of a finite point (or point at infinity)

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1) This does not hold even in the case of one variable. On the Riemann surface $\operatorname{Ln} z$ the minimal limiting distance of the region $1 < |z| < 2$ in the region $0 < |z| < 3$ is equal to unity, but the first of these regions does not lie entirely within the second region, since in this case the second condition in the definition of a region lying strictly inside another region would not be satisfied.

could under a continuous mapping be the point at infinity. Here the Weierstrass Theorems 1.2 and 1.3 on functions continuous in a closed region may be correctly applied only to finite continuous functions. The definition of uniform convergence remains the same as in the space C^n.

As was done in the Introduction, we here further introduce the concepts of manifold, surface (the definition presented there will be completely applicable here) and, in particular, of curve. By means of the concept of neighborhood we may in the usual way introduce the concept of interior point of a set of points of a covering region over the space P^n, and with the concept of a curve one may define a linearly connected set.

In concluding this section we shall discuss the problem of carrying over the Borel lemma on coverings to our regions.

Suppose that the set of points B of the region D has the property that every infinite subset of it has a limit point belonging to the set B. (In what follows the set B will be called *compact in itself* or simply compact.)

If to each point P of this set B one puts in correspondence some neighborhood $V_D(P)$ of P, then from the system of such neighborhoods one may always single out a finite collection of them such that the selected neighborhoods cover the entire set B.

Let us assume that this proposition is false. Then we consider the set of geometrical points $\underline{B}$ and now find in it a point $\underline{P}$ such that the collection of all points of the set B lying above any arbitrarily small neighborhood $\underline{V}$ of that point cannot be covered by a finite set of neighborhoods of our system.

On the other hand, from the assumption that the set B is compact in itself it follows that over the geometrical point $\underline{P}$ there lie only finitely many distinct analytic points of the set B. Indeed, if they form an infinite set, then the limit points of the latter must belong to the set B. However, from the definition of limit point it follows that if the limit point belongs to the set B, then in our case all the points of the sequence converging to it coincide with it with the possible exclusion of a finite set of them (since all the points of it have the same coordinates). In such a sequence there will therefore be only a finite set of distinct analytic points. Therefore our assertion easily follows.

If the number of these points is equal to s, then for a covering set of the part of the set B in question, s neighborhoods will obviously suffice. Thus we arrive at a contradiction and our proposition is proved.

4. **Canonical coverings and canonical covering regions.** Given a certain covering region D over the space P^n distinct from the whole space, let M_0 be some fixed finite point of that region. We consider an elementary neighborhood S_0 of the point M_0 in the region D. Then we choose a countable sequence of points $M_{\nu_1}(\nu_1 = 0, 1, 2, \cdots)$, everywhere dense in the cover S_0, and consider the regions S_{ν_1}, namely the elementary neighborhoods of the points M_{ν_1} in the region D. In each cover S_{ν_1} we again choose an everywhere dense sequence of points $M_{\nu_1\nu_2}$ (ν_1 fixed, $\nu_2 = 0, 1, \cdots$; $M_{\nu_1 0} \equiv M_{\nu_1}$) and for them we define elementary neighborhoods $S_{\nu_1\nu_2}$ in the region D. Continuing this process, we obtain a countable sequence of points $P_0, P_1, \cdots$, and a countable sequence of elementary neighborhoods $T_0, T_1, \cdots$. We define the number ϵ_{ij} for each pair T_i, T_j of these covers as equal to unity or zero, depending on whether or not there exists a point of D belonging to both covers T_i and T_j. We enumerate these covers in accordance with the requirement mentioned in the definition of region (see subsection 1 of the present section). Then we will say that the covers from the system $\{T_i, \epsilon_{ij}\}$ form a *canonical covering* of the region D. Evidently all the points of the region will be covered by covers of this system.

DEFINITION (canonical covering region). Suppose that the system of covers $\{S_i, \epsilon_{ij}\}$ defines a canonical covering of the region D. Every covering region defined by a system $\{S_i, \epsilon'_{ij}\}$, where $\epsilon'_{ij} \leq \epsilon_{ij}$ with the $<$ sign holding in at least one case, is said to be a canonical covering region for the region D.

A region Δ which is canonically covering for all regions canonically covering the region D is said to be a *universal covering* for D.

It follows from our definitions that a canonical covering region for D lies inside the covered region. Replacing some of the $\epsilon_{ij} = 1$ by the numbers $\epsilon'_{ij} = 0$, we increase the ramification of the region (we separate sheets previously sewn together), thus making distinct points which up to now were identical.

5. **Intersection of regions.** Suppose that we are given a finite or infinite set of regions E_k $(k = 0, 1, 2, \cdots)$, which respectively contain points $P^{(k)}$ with their neighborhoods $U^{(k)}$. These latter have one and the same geometrical point $\underline{P}$ and its neighborhood $\underline{U}$ as a projection. We consider the element of largest radius R_0 with center at the point $\underline{P}$ which is a fundamental region for the neighborhoods $S_0^{(k)}$ of the analytic points $P^{(k)}$ in all the regions E_k. We

denote this element by $\underline{S_0}$. Further, as above, we choose a countable sequence of points $\underline{P_{\nu_1}}$ lying everywhere dense in the region S_0. For each point $\underline{P_{\nu_1}}$ we consider the element $\underline{S_{\nu_1}}$ of largest radius which is a fundamental region for the neighborhoods $S_{\nu_1}^{(k)}$ of the analytic points in all the regions E_k.

Proceeding further in the usual way, we obtain a sequence of elements $\underline{S_i}$. We put $\epsilon_{ij} = 1$ if the covers $S_i^{(k)}$ and $S_j^{(k)}$ have a nonempty intersection for all k,[1] and in the contrary case we put $\epsilon_{ij} = 0$. The covering thus constructed $\{\underline{S_i}, \epsilon_{ij}\}$ defines a covering region D which we shall call *the intersection of the regions* E_k *relative to the points* $P^{(k)}$.

It follows from our construction that the region D lies inside all the regions E_k.

From continuity arguments it further follows that if there exists a region G lying inside all the regions E_k, then the intersections of the regions E_k relative to any two points P and Q of the region G (more precisely, relative to the points $P^{(k)}$ and $Q^{(k)}$ of the regions E_k, corresponding to the points P and Q; they always exist, since the region G lies inside all the regions E_k) turn out to be identical.

Indeed, joining the points P and Q by a broken curve with sufficiently many links so that the intersection regions with respect to successive vertices are identical, we arrive at the desired conclusion, passing from the point P to the point Q by successive steps along this broken curve.

In this case the region of intersection is called *the intersection of the regions* E_k *relative to the region* G.

6. Boundary points.

DEFINITION (boundary points of a covering region over the space P^n). An infinite sequence of points $\{P^k, k = 1, 2, \cdots\}$ of some region D, not having a limit point in this region, is said to be an (accessible) boundary point of the region D if 1) the sequence of fundamental points $\underline{P_k}$ converges to some point $\underline{R}$; 2) for each neighborhood $\underline{U_R}$ there exists a number $N > 0$ such that

1) Therefore a region obtained as an intersection turns out to be ramified to a degree corresponding to the most ramified of the intersecting regions. Analogously for the case of one variable; the intersection of the regions $D_1\{0 < |z| < 1\}$ on the Riemann surfaces $\sqrt{z}$ and $D_2\{0 < |z| < 2\}$ on the z-plane turns out to be D_1.

for $m_1, m_2 > N$ the points P_{m_1}, P_{m_2} may be joined in the region D by a curve whose projection lies in the neighborhood $U_{\underline{R}}$.

We will regard the boundary points R_1 and R_2 defined by the sequences $\{P_k\}$ and $\{Q_k\}$ as *identical* if $\lim \underline{P_k} = \lim \underline{Q_k} = \underline{R}$ and if for every neighborhood $U_{\underline{R}}$ there exists a number $N > 0$ such that for $m, n > N$ the points P_m and Q_n may be joined in the region D by a curve whose projection lies in the neighborhood $U_{\underline{R}}$.

A further, equivalent, definition of boundary point of a covering region may be given by using the concept of a filter of regions: a boundary point of the region D over the space P^n is a filter[1] of subregions $\{U\}$ of the region D with the following properties: 1) the regions $U \in R$ have no common points; 2) the closures of the fundamental regions $\underline{U}$ have one and only one common point $\underline{R} \in P^n$ and 3) each (sufficiently small in the sense of chordal distance) neighborhood $U_{\underline{R}}$ of the point $\underline{R}$ is a projection for one and only one region $U \in R$; each region $U \in R$ has such a neighborhood as its projection.

The set of all boundary points of the region D will be called the boundary of that region and denoted by ∂D. The set $D \cup \partial D$ will be called the extended region $\breve{D}$.

Every subregion of the region D containing an infinite set of points P_k used in the definition of the boundary point R will be said to be *adherent* to the point R.

Suppose that R is a boundary point of the region D. A subregion $V_D(R)$ of the region D adherent to the point R will be called a *neighborhood of the boundary point R in the region D* if 1) the closure of its fundamental region $\overline{V_{\underline{R}}}$ contains a neighborhood of the point $\underline{R}$; 2) its intersection with any subregion of the region D adherent to the point R contains a nonempty subregion of the region D which is adherent to the point R.

We adjoin to the neighborhood $V_D(R)$ those boundary points of the region D which 1) may be formed with the aid of a sequence of points of the region D lying in the neighborhood $V_D(R)$; 2) have projections lying in the closed region

1) By a filter Φ over some nonempty set M we mean a nonempty set of nonempty subsets of the set M such that for any two sets $W_1, W_2 \in \Phi$ there exists a set $W_3 \in \Phi$ such that $(W_1 \cap W_2) \supset W_3$.

$\overline{V_D(R)}$. Then we obtain the extended neighborhood $\breve{V}_D(R)$ of the boundary point $\underline{R}$ of the region D.

DEFINITION (branch point of a region). If every neighborhood of the boundary point R of the region D is non-single-sheeted, then the point R is said to be a branch point of the region D. If there exists a neighborhood $V_D(R)$ such that in any neighborhood $W_D(R) \subset V_D(R)$ there are always m points covering one and the same point $\underline{P} \in \underline{W_D(R)}$, and on the other hand no more than m points lie over any geometrical point, then the number m is said to be the order of the branch point R (here $1 \leq m \leq \infty$).

Evidently this defines the order of a branch point R uniquely.

Below, in subsection 4 of §10, we shall consider an important class of uniformized boundary points of a covering region.

We shall say that the sequence of points $P_k \in \breve{D}$ $(k = 1, 2, \cdots)$ converges to the boundary point R $(\lim P_n = R)$ if outside any neighborhood $V_D(R)$ of the point R there is only a finite set of points P_n of the region D and if only a finite set of boundary points belonging to the sequence P_n cannot be formed by means of points of the neighborhood $V_D(R)$. Further, with the aid of the concept of neighborhood of the boundary point R we may, in the obvious way, define a limit set $S_0 \subset D + \breve{\partial D}$ for the sequence of point sets $S_\nu \subset D$ $(\nu = 1, 2, \cdots)$, and give a definition of a function continuous at R, and of a mapping of the set $D \cup \breve{\partial D}$ onto another appropriately defined set. As usual, in such a case we consider only univalent functions and mappings.

The boundary points of a region D will be *finite* or *infinite*, depending on whether their projections are finite or infinite.

Suppose that the coverings of the system $\{S_i, \epsilon_{ij}\}$ define a canonical covering of the region D. Then on the boundary of each cover S_i there is at least one point not belonging to the region D. If those points on the boundary of the cover S_i form a finite or countable set, we include them all in our system. If they form a set of the power of the continuum, then we include in our system only a certain countable subset of this set which is everywhere dense in it. We choose such points on the boundaries of all the covers S_i. Here we shall consider two such points chosen on the boundaries of the covers S_i and S_j as identical if they have equal coordinates and $\epsilon_{ij} = 1$.

The set of points thus obtained is said to be *a canonical system of boundary points of the region D*. Evidently this canonical system consists of a finite or

countable set of points.

If the given region D is a subregion of some region Δ and the point $R \in \Delta$ is the boundary point for the region D, then the neighborhood $V_D(R)$ is a subregion of the corresponding neighborhood $U_\Delta(R)$.

If the region $D \ll \Delta$, then we may speak of the boundary of the region D as the set ∂D of points of the region Δ which are limiting for sequences of points of the region D and are not adherent to this set. The point set $D + \partial D$ is closed. We will denote it by $\overline{D}$ and call it a *closed region*. Here $\partial D = \breve{\partial D}$ if the boundary ∂D consists of accessible points. From now on we shall restrict ourselves to this case. We note that a finite-sheeted (in particular, one-sheeted) bounded region with the boundary adjoined to it always turns out to be closed. In the general case this does not hold (this is the situation also in the case of one variable: for example, the region $\{1 < |z| < 2\}$ on the Riemann surface $\operatorname{Ln} z$ is finite, but it does not become closed after adjunction to it of its boundary points).

§9. HOLOMORPHIC FUNCTIONS AND ANALYTIC SETS IN PLANE COVERING REGIONS. HOLOMORPHICITY REGIONS AND SINGULAR POINTS OF HOLOMORPHIC FUNCTIONS

1. Holomorphic functions and holomorphicity regions.

DEFINITION (functions holomorphic in covering regions over the space P^n). A function $f(z)$, defined in the plane region D over the space P^n, given by a system of covers $\{S_i, \epsilon_{ij}\}$, is said to be holomorphic in that region if in each cover S_i there may be defined a holomorphic function $f_i(\underline{z})$ such that 1) $f(z) = f_i(\underline{z})$ for $\underline{z} \in S_i$; 2) for $\epsilon_{ij} = 1$ the values of the functions f_i and f_j in the intersection of the regions S_i and S_j coincide.

The function $f(z)$ is said to be *holomorphic at the analytic point* z^0 if it is holomorphic in some region D to which that point z^0 belongs. Evidently a function f holomorphic in some region D necessarily is continuous and therefore univalent in that region.

If the region D_1 lies inside the region D and f is a holomorphic function defined in the region D by means of the system of holomorphic elements $\{f_i\}$, then these elements define a holomorphic function also in the region D_1. This function is usually denoted by the same symbol f.

Conversely, if the holomorphic function f_0 is given in the region $D_0 < D$, then in the region D there may not exist a holomorphic function f coinciding with the function f_0 at the corresponding points. The case when this holds is considered in the following definition:

DEFINITION (analytic continuation). If 1) the region D_0 lies inside the region D; 2) f_0 and f are holomorphic functions in the regions D_0 and D, while $f_0 = f$ for the corresponding points of these regions, then the function f is said to be the analytic continuation of the function f_0 from the region D_0 to the region D.

If the region D_0 is a subregion of the region D, the point $R \in D$ is a boundary point of the region D_0 and the function f_0 may be analytically continued to the region $D_1 \cup U_D(R)$, where $U_D(R)$ is some neighborhood of the point R in the region D, then one says that *the function* f_0 *may be continued analytically at the point* R. Sometimes such a boundary point R of the region D_0 in which the function f_0 was originally given is called a *removable singular point* of the function f_0.

Above we noted that a canonical covering region always is interior to the covered region (see subsection 4 of §8). This circumstance is completely consistent with the fact that every function holomorphic in the region D remains holomorphic in any region canonically covering the region D. For example, the function $f(z) = z$ is holomorphic not only in the ring $1 < |z| < 2$ on the z-plane, but also in the part of the Riemann surface $\sqrt{z}$ which lies above this ring. The converse does not always hold. For example, the function $f(z) = \sqrt{z}$ is holomorphic in the indicated ring on its Riemann surface, but it is not holomorphic in this ring in the z-plane. In fact, it is not univalent there. On the other hand, the function $f(z) = z$ is holomorphic in both ringshaped regions. We must consider the function $f(z) = z$, defined in the ring on the z-plane, as an analytic continuation of the function $f(z) = z$, considered in the ring on the Riemann surface $\sqrt{z}$.

In the same way as for regions of the space P^n (see §6 of the preceding chapter) we introduce for plane covering regions the concept of analytic continuation of the function f_0 from one subregion D_0 of the region D to another.

The process considered in §6 of the preceding chapter for analytic continuation of a holomorphic functional element f_0 generates, generally speaking, a multiple-valued analytic function F in some region $\underline{D} \subset P^n$. Now we shall

show that in our case that process may be supplemented by the simultaneous construction of some region D lying over that region $\underline{D}$. In the indicated region D we will define a holomorphic (i.e., univalent) function f whose values at the points $z \in D$ lying over the point $\underline{z} \in \underline{D}$ will coincide with the values of the function F at the point $\underline{z}$. This function f, in accordance with the definitions given above, will be an analytic continuation of the original holomorphic functional element f_0 to the region D.

As the initial element we choose an arbitrary holomorphic functional element f_0, given in the region $S(\underline{M}_0, R_0) = S_0$ (which may be of either the first or second kind) by a power series with center at the point M_0. This series has the form (1.41) if M_0 is a finite point, or else the form (1.77) if M_0 is a point at infinity. The radius R_0 of the region S_0 is taken to be maximal. Thus, on the boundary of the region S_0 there are points at which the series diverges.

Suppose that $\underline{M}_\nu$ ($\nu = 0, 1, 2, \cdots$) is some countable sequence of points, everywhere dense in the region S_0. We transform the given power series into a power series with center at the point M_{ν_1} (where $\nu_1 = 1, 2, \cdots$) and consider the cover $S(\underline{M}_{\nu_1}, R_{\nu_1}) = S_{\nu_1}$ of maximal radius R_{ν_1} in which that series converges. Then in each such cover S_{ν_1} we again choose an everywhere dense sequence of points $M_{\nu_1\nu_2}$ (where ν_1 is fixed, $\nu_2 = 0, 1, 2, \cdots$) and define for them a set of covers $S(\underline{M}_{\nu_1\nu_2}, R_{\nu_1\nu_2}) = S_{\nu_1\nu_2}$ (here $S_{\nu_1 0} = S_{\nu_1}$, $R_{\nu_1 0} = R_{\nu_1}$) of maximal radii $R_{\nu_1\nu_2}$ in which the power series obtained from the series with center at the point $\underline{M}_{\nu_1}$ as a result of this transformation converge. Further, we consider sequences of points $\underline{M}_{\nu_1\nu_2\nu_3}$ (where ν_1, ν_2 are fixed, $\nu_3 = 0, 1, 2, \cdots, \underline{M}_{\nu_1\nu_2 0} = \underline{M}_{\nu_1\nu_2}$) and the corresponding covers $S_{\nu_1\nu_2\nu_3}$, and so forth. Finally, we change the enumeration of the points $\underline{M}_{\nu_1}, \underline{M}_{\nu_1\nu_2}, \cdots$ and call them points $P_0, P_1, P_2, \cdots$, and their corresponding covers the regions T_0, T_1, T_2, and finally their corresponding holomorphic functional elements $f_0, f_1, f_2, \cdots$. We agree to take $\epsilon_{ij} = 1$ if the regions T_i and T_j have common geometrical points and the functional elements f_i and f_j defined in them are equal at these points. In other cases we put $\epsilon_{ij} = 0$.

Evidently the numbers ϵ_{ij} satisfy the conditions indicated in the definition of a system of covers. However, from the way in which the elementary regions T_i were obtained in the analytic continuation of the original functional element it turns out that it is possible to enumerate them in the way required by the

definition of a plane covering region over the space P^n (we leave to the reader the task of producing this enumeration). Thus the system of covers $\{T_i, \epsilon_{ij}\}$ defines a plane covering region D over the space P^n.

We define a holomorphic function $f(z)$ in the region D by putting $f(z) = f_i(\underline{z})$ for $\underline{z} \in T_i$. This function f is the analytic continuation of the holomorphic functional element f_0. In this connection we shall also call the function f complete analytic or simply analytic in the region D. However, as distinct from analytic functions in the region $\underline{D} \subset P^n$, considered in the preceding chapter, an analytic function $f(z)$ is single-valued in the region D.

The region D is said to be a *holomorphy region* (*domain of holomorphy*) of the analytic function f. This region is sometimes also called a *regularity region* or *region of existence* of the analytic function f.

From the way in which we constructed the region D to be a holomorphy region for the analytic function f, it results that this function cannot be analytically continued to a region $E > D$.

DEFINITION (holomorphy region). A covering region D over the space P^n is said to be a holomorphy region if there exists in it a holomorphic function f which cannot be analytically continued to some region $E > D$.

From the theorem of §6 of Chapter I we may, by the way, conclude that there exist regions (and moreover one-sheeted regions) which are not holomorphy regions.

We note that inasmuch as each holomorphy region may be constructed in the way described above, in each region which is a subregion of a holomorphy region there always exists a holomorphic function having at each two analytic points with the same coordinates distinct functional elements. These functional elements, being different, may have the same value at these points if the initial terms of the power series defining them coincide. Then we choose suitable arbitrary powers from these series so that their convergence regions do not change. With the new series we define in the region in question a holomorphic function which takes on distinct values at our points with the same coordinates.

The existence of such a function is what is meant by the property of *holomorphic separability* of a holomorphy region. Because of this property it is superfluous to introduce branch points into a holomorphy region.

We note further that on the boundary of each elementary region T_i used in

the construction of a holomorphy region for the function f, there is a point to which that function cannot be analytically continued. In the contrary case this elementary region T_i can be extended.

Functions holomorphic at some point z belonging to the plane covering region D form a domain of integrity $\mathfrak{O}_z$. For this ring Theorems 4.5 and 4.8 are valid, so that it is a Noether ring and we have the theorem on the uniqueness (up to equivalent factors) of the decomposition of the function $f \in \mathfrak{O}_z$ into the product of irreducible functions belonging to the ring $\mathfrak{O}_z$.

A ring of integrity is also formed by the functions holomorphic in some region $D_1 \subset D$. Functions holomorphic on some open set $B \subset D$ also form a commutative ring; however, it is generally speaking not a domain of integrity.

The collection of rings $\mathfrak{O}_z$ for various points $z \in D_1$ or $z \in B$ forms a bundle, usually denoted by the symbol $\mathfrak{O}(D_1)$ or $\mathfrak{O}(B)$.

In concluding this subsection we note the following definition.

DEFINITION (analytic set, sparse, almost sparse set). A subset m of the plane covering region D is said to be analytic in that region if each point $z \in D$ has a neighborhood $U_D(z)$ such that the set $m \cap U_D(z)$ coincides with the set of common zeros of some finite collection of functions holomorphic in that neighborhood $U_D(z)$.

A subset N of the region D is said to be sparse in that set if it is closed in D and if each point $z \in N$ has a neighborhood $U_D(z)$ such that the set $N \cap U_D(z)$ is contained in some analytic subset m_z of the neighborhood $U_D(z)$ which is nowhere dense in $U_D(z)$. The union of a countable set of sparse sets is said to be an almost sparse set.

For plane covering regions the Riemann theorem on analytic continuation of holomorphic functions holds (Theorem 6.7 of the preceding chapter).

2. Singular points of a holomorphic function.

DEFINITION. If in a covering region D over the space P^n there is defined a holomorphic function f which cannot be analytically continued to any region $E > D$, containing the region D as a subregion, then the boundary points of the region D are called nonremovable singular points of the function f.

In other words, the singular points of a holomorphic function are the boundary points of its holomorphy region or of a region canonically covering this holomorphy region.

The boundary points of a holomorphy region of a function constitute its *natural boundary*.

REMARK. Along with the nonremovable singular points discussed in our definition, a holomorphic function may have also the so-called removable singular points. Their definition was given in the preceding subsection.

We now distinguish a certain class of nonremovable singular points of holomorphic functions.

DEFINITION (of meromorphic functions). A function $f(z)$ is said to be meromorphic in a covering region D if:

1) the function is holomorphic in a set $D \setminus N$, where N is a sparse set in the region D;

2) the function cannot be analytically continued to any point of the set N;

3) for each point $\zeta \in N$ one may find a connected neighborhood $U_D(\zeta)$ and a function $q(z)$ holomorphic in that neighborhood and not identically zero there (but $q(\zeta) = 0$) such that the function $f(z)\, q_\zeta(z) = p_\zeta(z)$, which is holomorphic on the set $U_D(\zeta) \setminus N$, can be continued into the entire neighborhood $U_D(\zeta)$.

The set N is said to be the *polar set* of the function $f(z)$ in the region D. We have assumed it to be sparse, which is convenient in extending the definition of a meromorphic function to spaces of more general form. In reality it is always analytic, as we establish below. If the set N is empty, the function f is holomorphic in the region D.

The function f is said to be meromorphic at a point $z \in D$ if it is meromorphic in some neighborhood $V_D(z)$ of that point.

If the point $\zeta \in N$, then $q_\zeta(\zeta) = 0$. We may always suppose that the functions $p_\zeta(z)$ and $q_\zeta(z)$ in the neighborhood $V_D(\zeta)$ have no common holomorphic divisor equal to zero at the point ζ. If there were such a divisor for the original choice of the function $q(z)$ (it is discovered by applying to the functions $p(z)$ and $q(z)$ the algorithm for finding the greatest common divisor), we could get rid of it by cancellation. Then at the points $z \in V_D(\zeta)$, where $q_\zeta(z) \neq 0$, the function $f(z) = p(z)/q(z)$ is holomorphic. For the points $z \in V_D(\zeta)$, where $q_\zeta(z) = 0$ (in particular at the point ζ itself) two cases are possible:

1) $p_\zeta(z) \neq 0$; in this case z is a *pole* of the function f;

2) $p_\zeta(z) = 0$; in this case z is an *ambiguous point* of the function f.

In the neighborhood of a pole the modulus of the function f is unbounded.

In each neighborhood of an ambiguous point of the function f, it takes on any value α. It is equal to α on the analytic set $p - \alpha q = 0$, to which this ambiguous point necessarily belongs.

Thus, in the neighborhood $V_D(\zeta)$ of each point $\zeta \in N$ the set N coincides with the set of zeros of a holomorphic function $q_\zeta(z)$, which shows its analytic character. In view of Theorem 4.5 the function $q_\zeta(z)$ may be represented uniquely in the neighborhood $V_D(\zeta)$ in the form of a product

$$q_\zeta(z) = [q_\zeta^{(1)}(z)]^{p_1} \cdots [q_\zeta^{(k)}(z)]^{p_k}$$

Here $q_\zeta^{(s)}(z)$, $s = 1, \cdots, k$ are functions irreducible and holomorphic at the point ζ, $q_\zeta^{(s)}(\zeta) = 0$. We will call the analytic sets $q_\zeta^{(s)}(z) = 0$ $(s = 1, \cdots, k)$ irreducible polar sets of the function $f(z)$ passing through the point ζ, and the numbers p_s the orders of these sets for the function $f(z)$. Evidently the polar set N of the function $f(z)$ in the region D consists of such irreducible polar sets.

If $n = 2$, then in view of Theorem 4.1 the set of ambiguous points of a meromorphic function is discrete. In this case the singular points of the function lying in the neighborhood of the ambiguous point ζ are poles, filling out the analytic set defined by the equation $q_\zeta(z) = 0$. In the general case the complex dimension of the set of ambiguous points is less than the complex dimension of the entire polar set of the function by at least unity.

Consider the set $\mathfrak{M}_z$ of all functions meromorphic at some point $z \in D$. As we have just seen, from the definition of a meromorphic function it follows that the set $\mathfrak{M}_z$ is a quotient field for the ring $\mathfrak{D}_z$ consisting of the functions holomorphic at the point z.

Inasmuch as for the ring $\mathfrak{D}_z$ Theorem 4.5 on the uniqueness of the expansion of a function into holomorphic irreducible factors holds, this ring is integrally closed in its quotient field $\mathfrak{M}_z$.[1)]

Of course, the definition of meromorphic function given above may also be

1) See for example B. L. van der Waerden, *Modern algebra*, Part II, § 100, p. 77.

We recall that the ring $\mathfrak{D}$ is said to be integrally closed in its quotient field $\mathfrak{M}$ if every element $h = f/g \in \mathfrak{M}$ (where $f, g \in \mathfrak{D}$, $g \neq 0$), for which some polynomial of the form $W^r + c_1 W^{r-1} + \cdots + c_r$ vanishes (where $c_1, \cdots, c_r \in \mathfrak{D}$), belongs to $\mathfrak{D}$.

used without change in the case of regions or points (in particular, the points at infinity) of the space P^n or G^n.

DEFINITION (essentially singular point). A singular point of a holomorphic function which is not a meromorphic point for that function is said to an essentially singular point of that function.

Among the essentially singular points of a holomorphic function belong its branch points.

DEFINITION (branch point of a holomorphic function). A branch point of order m of the holomorphy region (existence region) of a holomorphic function is said to be a branch point of order m of that function.

Below (see subsection 4 of §10) we shall single out an important class of singular points of a holomorphic function, namely its uniformizable singular points.

3. Theorem on the continuous distribution of singular points of a holomorphic function. In distinction from the case of one complex variable, a holomorphic function of two or more complex variables cannot have isolated nonremovable singular points. This is one of the facts which determine the essential difference between the theory of analytic functions of several complex variables and the classical case of one variable.

THEOREM 9.1 (on the continuous distribution of singular points of holomorphic functions). *Suppose that the region D over the space C^n is a holomorphy region for the function $f(z)$. If for some $\epsilon > 0$*

1) *for all $\nu = 1, 2, \cdots$ the closed disks $\overline{S}_\nu \subset D$, where $\underline{S}_\nu = \{| z_1 - a_1 | < \epsilon$, $z_j = a_j^{(\nu)}$, $j = 2, \cdots, n\}$, $\lim_{\nu\to\infty} a_j^{(\nu)} = a_j$;*

2) *there exists $\lim_{\nu\to\infty} \overline{S}_\nu = \overline{S}_0 \subset D + \partial \breve{D}$, where $S_0 = \{| z_1 - a_1 | < \epsilon$, $z_j = a_j$, $j = 2, \cdots, n\}$;*

3) *the circle $\partial S_0 \subset D$;*

then also the disk $S_0 \subset D$.

Thus, if the disk S_0 contains singular points of the function $f(z)$ but the circle ∂S_0 does not, they cannot be contained in all the closed disks $\overline{S}_\nu$ $(\nu = 1, 2, \cdots)$ making up a sequence converging to $\overline{S}_0$.

In the (single-sheeted) holomorphy region $D \subset C^n$ the position of the points $z \in D$ is completely determined by their coordinates. In this case Theorem 9.1 may be formulated in the following way.

THEOREM 9.1_1. *Suppose $a \in C^n$ is a boundary point of the holomorphy region $D \subset C^n$ of the function $f(z)$. If for some $\epsilon > 0$ the disk $\{|z_1 - a_1| = \epsilon, z_j = a_j, j = 2, \cdots, n\} \subset D$, then one can find a number $\delta > 0$ such that on each closed disk $\{|z_1 - a_1| \leq \epsilon, z_j = b_j, j = 2, \cdots, n\}$, where $|b_j - a_j| < \delta$, there will be points not belonging to the region D.*

Theorem 9.1_1 is due to F. Hartogs [2, 3]. It in essentials remains valid if D is a region over the space C^n but has a single-sheeted subregion adherent to the point a.

We shall present a proof of this theorem due to F. Hartogs. For simplicity we restrict ourselves to the case of two variables and put $a_1 = a_2 = 0$.

By the hypotheses of the theorem the function $f(z)$ is holomorphic at all the points $P_\theta(\epsilon e^{i\theta}, 0)$, where $0 \leq \theta \leq 2\pi$. Then each point P_θ has a neighborhood $V_0(P_\theta) = \{|z_1 - \epsilon e^{i\theta}| < \rho_\theta, |z_2| < \rho_\theta\}$, in which the function $f(z)$ is holomorphic. We choose $\delta < \min \rho_\theta$. Evidently the function $f(z)$ is holomorphic also at the points of the set $\bigcup_{0 \leq \theta \leq 2\pi} \overline{V_\delta(P_\theta)}$, in particular at the points where $|z_1| = \epsilon, |z_2| \leq \delta$.

Now we make use of Theorem 6.3, taking $w = z_1$, $z = z_2$. The rôle of the region D is played by the disk $|z_1| < \epsilon$, of the region E by the disk $|z_2| < \delta$, and of the set $\{|z_1| = \epsilon, |z_2| \leq \delta\}$ by the collection of points $\partial D \times \overline{E}$.

We suppose that for some b, where $0 < |b| < \delta$, the closed disk $\{|z_1| \leq \epsilon, z_2 = b\} \subset D$. Then the function $f(z)$ will be holomorphic at the points of the region $\{|z_1| \leq \epsilon, |z_2 - b| \leq \mu\}$, where $\mu > 0$ is some sufficiently small number. We take the disk $\{|z_2 - b| < \mu\}$ as the region K of Theorem 6.3. All the hypotheses of that theorem are now satisfied, and thus the function $f(z)$ can be continued analytically to the entire bicylinder $\{|z_1| < \epsilon, |z_2| < \delta\}$, in particular to the point $(0, 0)$. This contradicts the hypotheses of Theorem 9.1_1. Our supposition turns out to be invalid and the Hartogs theorem in the case at hand is proved.

Theorem 9.1 admits far-reaching generalizations. One of them will be considered at the end of this section, and another in §11 of the present chapter.

4. Meromorphic continuations. In the sequel it will be convenient for us to formulate the concept of meromorphic continuation of a function. A holomorphic functional element given originally at some point P is said to be *meromorphically continuable* in the region D if the function obtained from it by analytic

continuation is meromorphic in the region D.

If for the originally given functional element the point P itself is a point of meromorphy (i.e., a pole or an ambiguous point), then we take as the starting point of the analytic continuation a point P^* in the neighborhood of P in which that functional element is holomorphic.

Meromorphic continuation along a piecewise smooth curve L is defined in the same way. We enclose L in a thin $2n$-dimensional tube $\tilde{L}$, which may be obtained for example by taking the union of the points of hyperdisks of radius ρ with centers on L. Then we consider, as above, the analytic continuation of the functional element given at the initial point of the curve L. Again, if the function obtained by this continuation turns out to be meromorphic in $\tilde{L}$, we call it a meromorphic continuation of the initial functional element along L.

DEFINITION (meromorphy region). A covering region D over the space P^n is said to be a meromorphy region if there exists in it a meromorphic function f which cannot be meromorphically continued to some region $E > D$.

In this case the region D is said to be a meromorphy region of the function $f(z)$. The boundary points of the region D or of a region canonically covering the region D are essential singular points of the function $f(z)$.

There naturally arises the question as to what extent the class of meromorphy regions differs from that of holomorphy regions. In the second part of this book it will be proved that the class of meromorphy regions and holomorphy regions in the space C^n are identical.

The initial point in the chain of arguments leading to this fact is the significant analogy with the theory of singular points of holomorphic functions discovered by E. E. Levi [1, 2]. He found that for essentially singular points exactly the same theorems hold as Theorems 9.1 and 9.1_1, namely those obtained by replacing the words "holomorphic function" and "singular point" by the words "meromorphic function" and "essentially singular point." Their proof is best prefaced by an auxiliary proposition (in which for simplicity we restrict ourselves to the case of two variables):

LEMMA. *Suppose that the function $f(z_1, z_2)$ is holomorphic in the closed bicylindrical region $\{k \le |z_1| \le K,\ |z_2| \le h\}$ and therefore is represented there by the Laurent series*

$$f(z_1, z_2) = \sum_{\nu=-\infty}^{\nu=\infty} g_\nu(z_2)\, z_1^\nu \tag{2.1}$$

(here $g_\nu(z_2)$ is a holomorphic function of z_2 in the disk $|z_2| < h$).

In order that there should exist a function meromorphic in the closed bicylinder $\{|z_1| \leq K, |z_2| \leq h\}$ and coinciding with the given function in the given region, it is necessary and sufficient that in the closed disk $|z_2| \leq h$ the following relations should be satisfied:

$$A_0(z_2)\, g_{\nu-l}(z_2) + A_1(z_2)\, g_{\nu-l+1}(z_2) + \cdots + A_l(z_2)\, g_\nu(z_2) = 0. \tag{2.2}$$

Here $\nu = -1, -2, \cdots$; $A_0(z_2), \cdots, A_l(z_2)$ are holomorphic functions not identically zero in the disk $|z_2| < h$.

PROOF. First we suppose that condition (2.2) is satisfied, and we consider the function

$$\phi(z_1, z_2) = A_0(z_2)\, z_1^l + A_1(z_2)\, z_1^{l-1} + \cdots + A_l(z_2).$$

We form the product $f(z_1, z_2)\, \phi(z_1, z_2) = \psi(z_1, z_2)$. This product is evidently holomorphic in the bicylindrical region $\{k \leq |z_1| \leq K, |z_2| \leq h\}$, and therefore may there be expanded in Laurent series. However, from relations (2.2), all the terms with negative powers in this expression cancel. The function $\psi(z_1, z_2)$ turns out to be representable in the bicylindrical region in question by a series converging in the entire closed bicylinder $\{|z_1| \leq K, |z_2| \leq h\}$. Thus in our case the functions $\phi(z_1, z_2)$, $\psi(z_1, z_2)$ are holomorphic in the entire closed bicylinder $\{|z_1| \leq K, |z_2| \leq h\}$. There

$$f(z_1, z_2) = \frac{\psi(z_1, z_2)}{\phi(z_1, z_2)}. \tag{2.3}$$

In accordance with the definition of a meromorphic function this means that $f(z_1, z_2)$ admits a meromorphic continuation to the entire region in which the representation (2.3) operates, i.e., to the entire bicylinder $\{|z_1| \leq K, |z_2| \leq h\}$. The sufficiency of condition (2.2) is proved.

Now we shall prove the necessity of this condition. Suppose that $(z_1^{(0)}, z_2^{(0)})$ is a nonessentially singular point of the function $f(z_1, z_2)$ lying in the closed bicylinder $\{|z_1| \leq K, |z_2| \leq h\}$. Then in the neighborhood of this point

$$f(z_1, z_2) = \frac{B(z_1, z_2)}{C(z_1, z_2)}, \tag{2.4}$$

where the functions $B(z_1, z_2)$ and $C(z_1, z_2)$ are holomorphic in this neighborhood

and $C(z_1^{(0)}, z_2^{(0)}) = 0$.

From the Weierstrass Preparation Theorem 4.1, in some neighborhood of the point $(z_1^{(0)}, z_2^{(0)})$ the function $C(z_1, z_2)$ may be represented as follows:

$$C(z_1, z_2) = (z_2 - z_2^{(0)})^r [(z_1 - z_1^{(0)})^n + \\ + E_1(z_2)(z_1 - z_1^{(0)})^{n-1} + \cdots + E_n(z_2)]\,\Omega(z_1, z_2), \tag{2.5}$$

where the function $\Omega(z_1, z_2)$ is holomorphic at the point $(z_1^{(0)}, z_2^{(0)})$ and does not vanish there, and the functions $E_s(z_2)$ are holomorphic for $z_2 = z_2^{(0)}$; $E_s(z_2^{(0)}) = 0$.

We shall show that $r = 0$. Suppose that $r \neq 0$. Then $C(z_1, z_2^{(0)}) \equiv 0$. We consider the function $f(z_1, z_2)$ at the points $P(\zeta, z_2^{(0)})$ of the closed disk $\{|\zeta| \leq K,\ z_2 = z_2^{(0)}\}$. In some region $\{|z_1 - \zeta| < \rho_P',\ |z_2 - z_2^{(0)}| < \rho_P\}$ of each point the function $f(z_1, z_2)$, by the hypotheses of the theorem, may be represented in the form of a quotient $B_P(z_1, z_2)/C_P(z_1, z_2)$ of holomorphic functions $B_P(z_1, z_2)$, $C_P(z_1, z_2)$.

Choose a point $P_t(\zeta_t, z_2^{(0)})$ such that the disks

$$U_t\{|z_1 - \zeta_t| < \rho_{P_t}'\},\quad t = 1, 2, \cdots,$$

overlap each other (the disk U_t with the disk U_{t+1}) and cover the disk $|z_1| \leq K$. Suppose that $\zeta_1 = z_1^{(0)}$. If the denominator $C_{P_1}(z_1, z_2^{(0)}) \equiv 0$ at the points of the disk U_1, then evidently the corresponding denominators will be equal to zero for all the U_t which intersect with U_1. Therefore in U_t we will always have $C_{P_t}(z_1, z_2^{(0)}) \equiv 0$. This contradicts the hypotheses of our theorem on the holomorphy of the function $f(z_1, z_2)$ in the closed bicylindrical region $\{k \leq |z_1| \leq K,\ |z_2| \leq h\}$. Thus we arrive at the conclusion that indeed in (2.5) $r = 0$.

Now from (2.5) it will follow that

$$C(z_1, z_2^{(0)}) = (z_1 - z_1^{(0)})^n\,\Omega(z_1, z_2^{(0)}). \tag{2.6}$$

This means that at the points $(z_1, z_2^{(0)})$, where $0 < |z_1 - z_1^{(0)}| < \rho$, the denominator $C(z_1, z_2^{(0)})$ will not vanish, and the function $f(z_1, z_2)$ is holomorphic. The points z_1 for which the points $(z_1, z_2^{(0)})$ are nonessentially singular points of the function $f(z_1, z_2)$ are isolated in the disk $|z_1| \leq K$. Therefore the

set of those points z_1 in the disk $|z_1| \leq K$ is finite.

Among the nonessentially singular points of the function $f(z_1, z_2)$ in the bicylinder $\{|z_1| \leq K, |z_2| \leq h\}$, only a finite set can be ambiguous points. Therefore there may exist only a finite set of values z_2' which are z_2-coordinates of ambiguous points. Suppose that $z_2^{(0)}$ is taken to be not one of these numbers, and that $z_1^{(1)}, \cdots, z_1^{(l)}$ are the z_1-coordinates of the poles $(z_1, z_2^{(0)})$ of the function $f(z_1, z_2)$. Near each such pole the function $f(z_1, z_2)$ may be represented in the form of a quotient (2.4), and the function $C(z_1, z_2^{(0)})$ in the form of an expression (2.6) where we need only replace $z_1^{(0)}$ by some $z_1^{(k)}$ of the numbers $z_1^{(1)}, \cdots, z_1^{(l)}$. We shall take each value $z_1^{(k)}$ a number of times equal to the multiplicity of that pole for the function of one complex variable $f(z_1, z_2^{(0)})$ (this will be the number n from the corresponding formula (2.6)). Suppose further that l is the number of the points z_1 in the closed disk $|z_1| \leq K$ of the $z_2 = z_2^{(0)}$ plane obtained in this way. In the neighborhood of each such point $(z_1^{(k)}, z_2^{(0)})$ the corresponding functions $C(z_1, z_2)$ may be represented by formula (2.5). Therefore it follows that the number l is the same for all values of z_2 for $|z_2 - z_2^{(0)}| < \eta_{z_2^{(0)}}$, where $\eta_{z_2^{(0)}}$ is an appropriately chosen number. Covering the entire closed disk $|z_2| \leq h$ (with the exclusion of the ambiguous point z_2') by such neighborhoods $|z_2 - z_2^{(0)}| < \eta$, we find that l has one and the same value at every point of that disk, with the exclusion of the ambiguous points z_2'.

We consider the functions $z_1^{(k)}(z_2)$ in the disk $|z_2| \leq h$ and from them form the elementary symmetric functions

$$S_k[z_1^{(2)}(z_2), \cdots, z_1^{(2)}(z_2)] = (-1)^{k+1} A_k(z_2).$$

These functions are bounded and holomorphic (since always $|z_1^{(k)}| \leq K$) in the neighborhood of all points of the disk $|z_2| \leq h$, with the exclusion of the points z_2'. Evidently these functions may be analytically continued to the entire closed disk $|z_2| \leq h$ (i.e., the z_2' turn out to be removable singular points for them). Thus the quantities $z_1^{(1)}(z_2), \cdots, z_1^{(l)}(z_2)$ turn out to be roots of the equation

$$\phi(z_1, z_2) = z_1^l + A_1(z_2) z_1^{l-1} + \cdots + A_l(z_2) = 0, \tag{2.7}$$

where the coefficients $A_0(z_2), A_1(z_2), \cdots, A_l(z_2)$ are holomorphic in the entire disk $|z_2| \leq h$ (we have $A_0(z_2) \equiv 1$, so that it is evident that not all the $A_k(z_2)$

are identically zero).

We form the function

$$\psi(z_1, z_2) = \phi(z_1, z_2) f(z_1, z_2). \tag{2.8}$$

It follows from what has been said that $\psi(z_1, z_2)$, considered for each z_2 on the disk $|z_2| \leq h$ as a function of one variable z_1 in the disk $|z_1| \leq K$, can have there only removable singular points and will be represented by an entire series. After substitution in (2.8) of the series (2.1) in place of $f(z_1, z_2)$ and of the expression (2.7) in place of $\phi(z_1, z_2)$, all the terms with negative powers of z_1 will remain unchanged. Consequently, we arrive at conditions (2.2). The necessity of these conditions is thus proved, and the proof of our theorem completed.

REMARK. From our proof it follows that in formula (2.2) the quantity $A_0(z_2)$ may always be assumed identically equal to unity.

5. Theorem on the continuous distribution of essentially singular points of holomorphic functions. This theorem in the simplest case of a one-sheeted meromorphy region was proved by E. E. Levi [1].

THEOREM 9.2 (on the continuous distribution of singular points of holomorphic functions). *Suppose that the region D over the space C^n is a meromorphy region for the function $f(z)$. If for some $\epsilon > 0$:*

1) *for all $\nu = 1, 2, \cdots$ the closed disks $\overline{S}_\nu \subset D$, where $\underline{S}_\nu = \{|z_1 - a_1| < \epsilon$, $z_j = a_j^{(\nu)}$, $j = 2, \cdots, n\}$, $\lim_{\nu\to\infty} a_j^{(\nu)} = a_j$;*

2) *there exists the limit $\lim_{\nu\to\infty} \overline{S}_\nu = \overline{S}_0 \subset D + \partial\breve{D}$, where $\underline{S}_0\{|z_1 - a_1| < \epsilon$, $z_j = a_j$, $j = 2, \cdots, n\}$;*

3) *the circle $\partial S_0 \subset D$;*

then also the disk $S_0 \subset D$.

In the (single-sheeted) meromorphy region $D \subset C^n$ the position of the points $z \in D$ is completely determined by their coordinates. In this case Theorem 9.2 may be formulated in the following way.

THEOREM 9.2$_1$. *Suppose that $(a_1, \cdots, a_n) \in C^n$ is a boundary point of the meromorphy region $D \subset C^n$ for the function $f(z)$. If for some $\epsilon > 0$ the circle $\{|z_1 - a_1| = \epsilon$, $z_j = a_j$, $j = 2, \cdots, n\} \subset D$, then it is possible to find a number $\delta > 0$ such that on every closed disk $\{|z_1 - a_1| \leq \epsilon$, $z_j = b_j$, $j = 2, \cdots, n\}$, where $|b_j - a_j| < \delta$, there will exist points not belonging to the region D.*

Theorem 9.2_1 remains essentially valid if D is a region over the space C^n but has a single-sheeted subregion adherent to the point a.

We restrict ourselves to the proof of Theorem 9.2_1 for the case of two variables, and we put $a_1 = a_2 = 0$.

By hypothesis the function $f(z)$ is meromorphic at all points $P_\theta(\epsilon e^{i\theta}, 0)$, where $0 \leq \theta < 2\pi$. In some neighborhood U_θ of each point P_θ this function may be represented in the form of the quotient of two holomorphic functions. The points $\{P_\theta\}$ form a closed set. Therefore we may choose from them n points $(\zeta_k, 0)$, $\zeta_k = \epsilon e^{i\theta_k}$, $k = 1, \cdots, n$, such that their neighborhoods $U_{\theta_k}\{|z_1 - \zeta_k| < \rho,\ |z_2| < \rho'\} = U_k$ cover the entire circle $\{|z_1| = \epsilon,\ z_2 = 0\}$. We consider those neighborhoods U_k in which the function $f(z_1, z_2)$ is not a holomorphic function. There

$$f(z_1, z_2) = \frac{B_k(z_1, z_2)}{C_k(z_1, z_2)} \tag{2.9}$$

and in view of the Weierstrass Preparation Theorem 4.1 (we have $C_k(\zeta_k, 0) = 0$),

$$C_k(z_1, z_2) = z_2^{s_k}[(z_1 - \zeta_k)^{m_k} + A_1(z_2)(z_1 - \zeta_k)^{m_k - 1} + \cdots$$
$$\cdots + A_{m_k}(z_2)]\,\Omega(z_1, z_2). \tag{2.10}$$

Suppose that $N = \max(s_1, \cdots, s_n)$. We choose instead of the function $f(z_1, z_2)$ the function $f_1(z_1, z_2) = z_2^N f(z_1, z_2)$. Evidently all the s_k for the function $f_1(z_1, z_2)$ are equal to zero, which is easily verified if one replaces $f(z_1, z_2)$ in (2.9) by $z_2^{-N} f_1(z_1, z_2)$ and then considers the expansion of the denominator for the function $f_1(z_1, z_2)$, analogous to (2.10). Further it is clear that the essentially singular points of the functions $f(z_1, z_2)$ and $f_1(z_1, z_2)$ coincide, and therefore we may take $f_1(z_1, z_2)$ in place of $f(z_1, z_2)$. We denote $f_1(z_1, z_2)$ anew by $f(z_1, z_2)$ and further, in equations (2.10) we will take $s_k = 0$.

The number ρ may be chosen so that for $0 < |z_1 - \zeta_k| \leq \rho$ the functions $C_k(z_1, 0)$ do not vanish (to this end it may be necessary to increase the number of neighborhoods U_k). Then one may consider it as established that all essentially singular points of $f(z_1, z_2)$ lying in some ring $\{\epsilon - \eta \leq |z_1| < \epsilon + \eta,\ z_2 = 0\}$ are among the points ζ_k. Therefore at the points of the circle $\{|z_1| = \gamma,\ z_2 = 0\}$, where $\epsilon - \eta < \gamma < \epsilon + \eta$, $\gamma \neq \epsilon$, the function $f(z_1, z_2)$ is holomorphic. This means that the function $f(z_1, z_2)$ is holomorphic at all the points (z_1, z_2)

for $\gamma_1 \le |z_1| \le \gamma_2$, $|z_2| \le \delta$. (Here γ_1, γ_2 and δ are chosen appropriately.) Suppose that $z_2^{(0)}$ is chosen so that $|z_2^{(0)}| < \delta$. We suppose that our theorem is false and assume that the function $f(z_1, z_2)$ can be meromorphically continued to all points $(z_1, z_2^{(0)})$, where $|z_1| \le \gamma_2$. Then evidently the function $f(z_1, z_2)$ will be meromorphic also at the points (z_1, z_2), if $|z_1| \le \gamma_2$ and $|z_2 - z_2^{(0)}| \le \mu$. Here μ is an appropriately chosen positive number. We choose μ so that the disk $|z_2 - z_2^{(0)}| \le \mu$ lies inside the disk $|z_2| < \delta$. Then the function $f(z_1, z_2)$ will be holomorphic in the bicylindrical region $\{\gamma_1 \le |z_1| \le \gamma_2, |z_2 - z_2^{(0)}| \le \mu\}$ and may be meromorphically continued to the bicylindrical region $\{|z_1| \le \gamma_2, |z_2 - z_2^{(0)}| \le \mu\}$. Thus, all the hypotheses of the lemma of the preceding subsection on meromorphic continuation are satisfied. Because of this lemma, if we represent $f(z_1, z_2)$ in the bicylindrical region $\{\gamma_1 \le |z_1| \le \gamma_2, |z_2 - z_2^{(0)}| \le \mu\}$ by the Laurent series

$$f(z_1, z_2) = \sum_{\nu=-\infty}^{\nu=\infty} g_\nu(z_2)\, z_1^\nu \tag{2.11}$$

(where the functions $g_\nu(z_2)$ are holomorphic in the disk $|z_2 - z_2^{(0)}| \le \mu$), then the following relations will be satisfied:

$$A_0(z_2)\, g_{\nu-l}(z_2) + A_1(z_2)\, g_{\nu-l+1}(z_2) + \cdots + A_l(z_2)\, g_\nu(z_2) = 0. \tag{2.12}$$

Here $\nu = -1, -2, \cdots$, and the functions $A_0(z_2), A_1(z_2), \cdots, A_l(z_2)$ are holomorphic in the disk $|z_2 - z_2^{(0)}| \le \mu$ and are not all identically zero. On the other hand, the function $f(z_1, z_2)$ is holomorphic in the bicylindrical region $\{\gamma_1 \le |z_1| < \gamma_2, |z_2| \le \delta\}$ and in this region may be represented by a Laurent series of the form (2.11). Since the disk $|z_2 - z_2^{(0)}| \le \mu$ is a portion of the disk $|z_2| < \delta$, the Laurent series representing the function $f(z_1, z_2)$ in the entire bicylindrical region $\{\gamma_1 \le |z_1| \le \gamma, |z_2| \le \delta\}$ must coincide with the series (2.11) for $|z_2 - z_2^{(0)}| \le \mu$. This means that the function $g_\nu(z_2)$ can be analytically continued to the disk $|z_2| \le \delta$ and the expansion (2.11) will give the function $f(z_1, z_2)$ in the entire bicylindrical region $\{\gamma_1 \le |z_1| \le \gamma_2, |z_2| \le \delta\}$.

We now consider an infinite system of linear equations

$$\xi_0\, g_{\nu-l}(z_2) + \xi_1\, g_{\nu-l+1}(z_2) + \cdots + \xi_l\, g_\nu(z_2) = 0. \tag{2.13}$$

From (2.12) the equations (2.13) in the disk $|z_2 - z_2^{(0)}| \le \mu$, which is a portion of the disk $|z_2| \le \delta$, are satisfied by the quantities

$$\xi_k = A_k(z_2), \quad k = 0, 1, 2, \cdots, l,$$

which are not identically equal to zero. This means that the number of independent equations of the system (2.13) in the disk $|z_2 - z_2^{(0)}| \le \mu$ will be $\le l$. But this situation is expressed by relations between the quantities $g_\nu(z_2)$ (they will consist of the identical vanishing of a series of determinants made up from the quantities $g_\nu(z_2)$), which, because of the holomorphy of the functions $g_\nu(z_2)$ will be satisfied in all of this disk $|z_2| \le \delta$. Therefore it follows that in the disk $|z_2| \le \delta$ there will exist a system of solutions of the equations (2.13) which are not all identically zero, and thus the functions $g_\nu(z_2)$ satisfy there relations of the form (2.12). From our lemma one may conclude that the function $f(z_1, z_2)$ can be meromorphically continued to the entire closed bicylinder $\{|z_1| \le \epsilon, |z_2| \le \delta\}$ and in particular to the point (0, 0). This contradicts the hypothesis of the theorem, and we must take our supposition to be invalid.

Our theorem is proved.

6. Generalization of the theorems on continuous distribution of singular and essentially singular points. The analytic planes considered in Theorems 9.1_1 and 9.2_1 may be replaced by analytic surfaces. As a result one may obtain essential generalizations of these theorems.

DEFINITION. Consider in the space C^n of variables $z_1, \cdots, z_n$ a family of one-complex-dimensional analytic surfaces

$$(E_\alpha) \quad \phi_j(z_1, \cdots, z_n, \alpha_2, \cdots, \alpha_n) = 0, \; j = 2, \cdots, n.$$

Here $(\alpha_2, \cdots, \alpha_n)$ is a point of some region G of the space of complex variables $\alpha_2, \cdots, \alpha_n$. For each point $\alpha^0 \in G$ ϕ_j are holomorphic functions of the variables z in the neighborhood $U_a \subset C^n$ of each point $a \in E_{\alpha^0}$. For each point $z \in U_a$ ϕ_j are holomorphic functions of the variables α in some neighborhood V_{α^0} of the point α^0.[1)]

The family $\{E_\alpha\}$ is said to be *regular* in the neighborhood U_a of some point $a \in C^n$ if:

1) There exists an $\alpha = \alpha^0$ such that $a \in E_{\alpha^0}$;

1) From the Hartogs' Theorem 1.6 it therefore follows that the functions ϕ_j in the corresponding regions $U \times V$ are holomorphic in the set of their variables.

2) through each point $z \in U_a$ there passes one and only one surface of the family $\{E_\alpha\}$ (this will be so if the Jacobian $(\partial\phi/\partial\alpha)_{a,\alpha^0} \neq 0$, i.e., if the equations of the surfaces of the family $\{E_\alpha\}$ may in some neighborhood $U_a \times V_{\alpha^0}$ of each point (a, α^0) be written in the form

$$(E\alpha/_{a,\alpha^0}) \quad \psi_j(z_1, \cdots, z_n \mid a, \alpha^0) = \alpha_j, \; j = 2, \cdots, n);$$

3) the surfaces E_α consist of ordinary points.

The following theorem holds:

THEOREM 9.3. *Suppose that $a \in C^n$ is a boundary point of the region $D \subset C^n$ of holomorphy or meromorphy of the function $f(z)$, $\{E_\alpha\}$ is a family of analytic surfaces regular in some neighborhood U_a of the point a. If the point $a \in E_{\alpha^0}$ and $[(E_{\alpha^0} \cap U_a) \setminus a] \subset D$, then there exists a neighborhood V_{α^0} of the point α^0 such that for each point $\alpha \in V_{\alpha^0}$ the set $U_a \cap E_\alpha$ contains points not belonging to the region D.*

PROOF. Write the equations of the surfaces of our family in the form $E\alpha/_{a,\alpha^0}$. In accordance with condition 3) on the regularity of this family we suppose that $[\partial(\psi_2, \cdots, \psi_n)/\partial(z_2, \cdots, z_n)]_{z=a} \neq a$. Consider the mapping

$$(T) \qquad w_1 = z_1 - a_1; \quad w_j = \psi_j(z \mid a, \alpha^0) - \alpha_j^0, \; j = 2, \cdots, n,$$

of the region $U_a \subset C^n$ into the space of complex variables w. This mapping is biholomorphic, since its Jacobian $(\partial w/\partial z)_a = (\partial(\psi_2, \cdots, \psi_n)/\partial(z_2, \cdots, z_n))_a \neq 0$. As a result of the mapping T we obtain a function $F(w) = f(z)$. The point a goes under the mapping T into the origin of coordinates and the surface $E_\alpha \cap U_a$ goes into the plane $w_j = \alpha_j - \alpha_j^0$, $j = 2, \cdots, n$. Then for the holomorphy (or meromorphy) region D^* of the function $F(w)$ all the assumptions of Theorem 9.1_1 (or 9.2_1) are satisfied. We must take it as established that on each plane $w_j = \text{const.}$, $j = 2, \cdots, n$, there are in some neighborhood of the origin points not belonging to the region D^*. Turning by means of the mapping T^{-1} from the function $F(w)$ to the function $f(z)$, we find that the assertion of Theorem 9.3 is indeed satisfied.

§10. MAPPINGS OF REGIONS OVER THE SPACE P^n. INTERIOR-BRANCHED REGIONS

1. Holomorphic mappings of regions over the space P^n are special cases of continuous mappings on them.

DEFINITION (holomorphic mappings of regions over the space P^n). A continuous mapping $T\{w = w(z)\}$ of the region D over the projectively extended space P^n of variables $z_1, \cdots, z_n$ on the region D^* over the space C_w^m of variables $w_1, \cdots, w_m$ is said to be *holomorphic* if it puts into correspondence with each function $\phi(w)$ holomorphic in some subregion D_1^* of the region D^* the function $\phi \circ T = \phi(w(z))$ holomorphic in the subregion D_1 of the region D. Here the region $D_1 = T^{-1} D_1^*$.

If the region D^* is a subregion of the region $\mathfrak{D}^*$ over the space C_w^m, then we say that T is a holomorphic mapping of the region D into the region $\mathfrak{D}^*$. The mapping $D \to D^*$ is sometimes called a *surjective holomorphic* mapping. If E is a subregion of the region D, then the mapping T restricted to the points of the region E is said to be a *restriction* of that mapping and is denoted by the symbol $T|_E$.

The definitions given here are easily extended to the case of open sets over the space P^n.

The mapping T of the region D is said to be *holomorphic at the point* $z \in D$ if there exists a subregion $D_1 \ni z$ of the region D in which that mapping is holomorphic.

REMARK. From what has been said it follows that in the case being analyzed all the functions $w_k(z)$ (where w_k are the respective coordinates) are holomorphic in the region D, since one may always put $\phi(w) = w_k$ $(k = 1, \cdots, m)$. It is easy to see that in the case when the region D^* is single-sheeted, the holomorphy of the functions $w_k(z)$ $(k = 1, \cdots, m)$ is sufficient for the holomorphy of the mapping $w = w(z)$.

As the simplest example of a holomorphic mapping we have the relation $z \to \underline{z}$, where $z \in D$, $\underline{z} \in \underline{D}$, D is a region over the space P^n. We note that the inverse relation, under which the image of the original region $\underline{D}$ (the region D) is not single-sheeted, is not even univalent.

We turn to the consideration of the case when $m = n$. We will agree to say that the mapping $T\{w = w(z)\}$ of the region D over the space C_z^n into the region D^* over the space C_w^n is *biholomorphic* or *pseudoconformal* if both it itself and its inverse mapping T^{-1} of the region $T(D)$ onto the region D are holomorphisms. We note that holomorphic mappings of infinite regions cannot be biholomorphic in our sense, since the inverse mappings are certainly not holomorphisms.

Analogously one introduces the concept of mappings which are biholomorphic

at some point $z \in D$.

From the definition of a holomorphic mapping it results that a biholomorphic mapping is a homeomorphism.

Now suppose that in some region D over the space C^n of variables $z_1, \cdots, z_n$ we are given holomorphic functions $w_k = w_k(z)$ $(k = 1, \cdots, n)$. Then they put into correspondence with each point $z \in D$ some point $\underline{w}$ of the space C_w^n of variables $w_1, \cdots, w_n$. Its coordinates will be defined from the equations

$$w_k = w_k^{(i)} (z_1, \cdots, z_n), \; k = 1, \cdots, n, \; i = 0, 1, \cdots. \tag{2.14}$$

Here the index i takes on values which indicate the number of the elementary region $\sigma_i \ni z$ and the corresponding functional element of the holomorphic function $w_k(z)$. One may always suppose that this number is one and the same for all functions $w_k(z)$ for $k = 1, \cdots, n$. We denote the set of such points $\underline{w}$ by $\underline{D}^*$. Then the following theorem turns out to be true. It is essentially complementary to Theorem 7.1.

THEOREM 10.1. *If the Jacobian* $\partial w / \partial z = \partial(w_1^{(i)}, \cdots, w_n^{(i)})/\partial(z_1, \cdots, z_n)$, $i = 0, 1, \cdots$, *computed from equations* (2.14), *does not vanish in the region* D, *then over the set* $\underline{D}^*$ *(which in this case is a region) one may find a covering region* D^* *such that the relation* (2.1) *defines a biholomorphic mapping of the region* D *onto that region* D^*.

PROOF. Suppose that z is any point of the region D. Then from Theorem 7.1 there exists a neighborhood of this point U_z in the region D such that the functions (2.14) biholomorphically map the region $\underline{U}_z$ onto some neighborhood $\underline{V}_w$ of the point $\underline{w}$ $(w_1^{(i)}(z), \cdots, w_n^{(i)}(z))$. Here the number i is chosen so that $z \in \sigma_i$, and the neighborhood U_z so that the region V_w is a polycylinder $S(\underline{w}, r)$ with center at the point w and of radius r.

Now we choose as initial point $z_0 \in D$ and a neighborhood U_{z_0}. Suppose that to them correspond a point $\underline{w}_0$ and a polycylinder $S(\underline{w}_0, r_0) = S_0$. We choose in the latter a countable everywhere dense set of points $\underline{w}_i$ corresponding to the points z_i of the neighborhood U_{z_0} (forming there also an everywhere dense set). Each point $\underline{w}_i$ will be the center of a polycylinder S_i, corresponding to the neighborhood U_{z_i} of the point z_i. Then in each polycylinder S_i we choose an everywhere dense set of points $\underline{w}_{ij}$. To these there will correspond points z_{ij} of the neighborhood U_{z_i} and so forth. Operating in this manner, we obtain a

sequence of polycylinders $\{S_p\}$. We define for them constants ϵ_{pq} equal to 1 or to zero depending on whether the regions U_p and V_q have common analytic points. Then the collection $\{S_p, \epsilon_{pq}\}$ defines a finite covering region D^*. The mapping (2.14) of the region D onto the region D^* is holomorphic and one-to-one.

From Theorem 7.1 it further follows that the functional elements

$$z_k = z_k^{(p)}(w_1, \cdots, w_n), \quad k = 1, \cdots, n; \quad p = 0, 1, \cdots, \tag{2.15}$$

defined by the mapping indicated above of each polycylinder S_p onto the corresponding region U_p, are holomorphic in these polycylinders. The functions $z_k^{(p)}(w)$ and $z_k^{(q)}(w)$, defined in the elementary regions S_p and S_q, coincide with each other in the intersection if $\epsilon_{pq} = 1$ (which follows from the method of choice of the ϵ_{pq}). Thus, the holomorphic elements $z_k^{(p)}(w)$, defined in the elementary regions S_p, are analytic continuations of one of them and define in the region D^* a holomorphic function $z_k(w)$. Now if we also take into account the univalent character of the mapping (2.15) of the region D^* onto the region D, it is evident that this mapping is a holomorphism. The assertion of the theorem is proved.

From Theorem 7.3 it is evident that if at some point $z \in D$ the Jacobian $\partial w/\partial z = 0$, then one cannot define a region D^* over the space of variables $w_1, \cdots, w_n$ such that on it the functions (2.14) would define a holomorphic mapping of the region D.

2. Mappings with zero Jacobians. Consider again a region D over the space C_z^n of variables $z_1, \cdots, z_n$, the functions $w_k = w_k(z)$ $(k = 1, \cdots, n)$ holomorphic in this region, and the set $\underline{D}^*$ of points $\underline{w}$ of the space C_w^n of variables $w_1, \cdots, w_n$ which these functions put into correspondence with the points $z \in D$. The coordinates of these points $\underline{w}$ will be defined, as before, from equations (2.14). We denote by τ the mapping of the region D onto the region $\underline{D}^*$ thus defined. Suppose that the Jacobian $\partial w/\partial z$ of the functions (2.14) is not identically equal to zero in the region D but vanishes in that region on some set E. Suppose that D_0 is the set of points of the region D which remain after the exclusion of the points of the set E.

We shall show that the set D_0 is a region. Indeed, it is evident that if at the point $z \in D$ the Jacobian $\partial w/\partial z \neq 0$, then it differs from zero also in some neighborhood of the point z, which thus also belongs to the set D_0. The connectivity of the region cannot be violated by the exclusion from the region D of

points for which the Jacobian $\partial w/\partial z = 0$, since these points form one or several $(n-1)$-complex-dimensional analytic surfaces. From Theorem 10.1, by means of the functions (2.15) one can define (together with the mapping τ of the region D onto some region $\underline{D}^*$ of the space C^n_w of variables $w_1, \cdots, w_n$) a biholomorphic mapping T of the region D_0 onto some region D_0^* over that space C^n_w. The points of the set E belong to the frontier of the region D_0. Consider the result of application of the mapping τ to the points of this set. Its image evidently will be a certain set of geometric points $\underline{E}^* \subset C^n_w$, over which there are situated (defined in accordance with the statements in subsection 6 of §8) boundary points of the region D_0^*. Suppose for example that the Jacobian $\partial w/\partial z = 0$ is zero at the point $a \in D$ and that the point $w = \underline{b} \in \underline{E}^*$ is the image of the point a under the mapping (2.14). Then two cases are possible:

CASE 1. In some polycylinder $U_a = S(a, \epsilon)$ of radius ϵ, except for the point a there are no points $z \in \tau^{-1}(\tau a)$, i.e., points for which $\tau z = \underline{b}$.

CASE 2. In any neighborhood of the point a there is another point $z \in \tau^{-1}(\tau a)$, i.e., a point for which $\tau z = \underline{b}$.

Let us consider these cases in more detail.

CASE 1. Select an element σ_i of the region D containing the point a and the corresponding functions (2.14). Because of our hypotheses, the conditions of Osgood's Theorem 7.2 (see subsection 2 of §7) are satisfied for them at the point $\underline{z} = \underline{a}$. Because of this theorem the system of equations (2.1) for an appropriate index i has, in the neighborhood $\underline{U}_a$ of the point $\underline{z} = \underline{a}$, m solutions:

$$(\tau^{-1})\, z_k^{(i)(s)} = z_k^{(i)(s)}(w_1, \cdots, w_n); \quad k = 1, \cdots, n; \quad s = 1, \cdots, m, \tag{2.16}$$

defining in some neighborhood $\underline{V}_b = \tau \underline{U}_a$ of the point $\underline{w} = \underline{b}$ a mapping τ^{-1} inverse to the mapping τ. Here m is the degree of the corresponding pseudopolynomial (1.93), while in the case at hand $m > 1$. To each point $\underline{w} \in \underline{V}_b$ not belonging to the set $\underline{E}^*$ [1] the mapping τ^{-1} assigns m distinct points

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1) It is easy to see that inside the neighborhood $\underline{V}_b$ the set E^* coincides with the discriminant set Δ of the corresponding pseudopolynomial (1.93). Indeed, the set Δ is defined by the equations $\Gamma(z_2, w_1, w_2) = 0$, $\Gamma'_{z_2}(z_2, w_1, w_2) = 0$ (we restrict ourselves to the case of two variables, dropping the index i). Then from relations (2.1) we have at the points $z \in \tau\Delta \subset U_a$

$z^s(\underline{w})$ $(s = 1, \cdots, m)$ of the region D_0. Hence it follows that (in view of the one-to-one biholomorphic mapping T of the region D_0 onto the region D_0^*) over the geometric point $\underline{w}$ there lie m distinct points of this region D_0^*. As the point $\underline{w}$ tends to the point $\underline{b}$ the points $z^s(w)$ tend to coincide with one another. Therefore, it follows that over the point $\underline{b}$ there lies one boundary point $\underset{\sim}{b}$ of the region D_0^*. In accordance with the definition of subsection 6 of §8, this point $\underset{\sim}{b}$ is an m-tuple branch point of the region D_0^*. Reasoning in the same way we define a set E^* of branch points of the region D_0^* lying over the set $\underline{E}^*$.

It is easy to see that if the set $\underline{E}^*$ is connected, then all the branch points of the region D_0^* belonging to it have one and the same order m. This follows from the fact that the order is an integer and is a continuous function of the position of the point w. If the set E^* is decomposed into several connected components, then the branch points of the region D_0^* belonging to the same component of the set E^* have the same branching order.

We continue the mapping T to the points of the set E, by putting $b = Ta$ for the points $a \in E$, so that $E^* = TE$. Consider the extended neighborhood $\tilde{V}_b(D_0^*)$ of the boundary point b in the region D_0^*. From our considerations it follows that as a result of its continuation T becomes a homeomorphic mapping of the neighborhood U_a of the point $z = a$ onto that extended neighborhood $\tilde{V}_b(D_0^*)$ of the point b.

The functions $z_k^{(s)}(w)$ $(k = 1, \cdots, n;\ s = 1, \cdots, m)$ cannot be holomorphic for all k in the entire neighborhood V_b and, in particular, at the point $w = b$ (in the contrary case they must coincide with one another for all s). By the definition given in subsection 2 of §9, the point $w = b$ is a branch point for

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$$\frac{\partial\Gamma}{\partial w_1}\frac{\partial w_1}{\partial z_1} + \frac{\partial\Gamma}{\partial w_2}\frac{\partial w_2}{\partial z_1} = 0; \quad \frac{\partial\Gamma}{\partial z_2} + \frac{\partial\Gamma}{\partial w_1}\frac{\partial w_1}{\partial z_2} + \frac{\partial\Gamma}{\partial w_2}\frac{\partial w_2}{\partial z_1} = 0.$$

Since at the point $\underline{w} \in \Delta$ the derivative $\Gamma'_{z_2} = 0$, at the corresponding point $z \in r^{-1}\Delta$ the Jacobian $\partial w/\partial z = 0$. Therefore it follows that that point $z \in E$, and the point $\underline{w} = rz \in E^*$. The case when the point (z_2, w_1, w_2) (where w_1, w_2 are the coordinates of the point $\underline{w}$, z_2 the coordinate of the point $z = r^{-1}\underline{w}$) is a singular point of the surface $\Gamma = 0$ in the six-dimensional space of the variables z_2, w_1, w_2 (in it $\Gamma'_{z_2} = \Gamma'_{w_1} = \Gamma'_{w_2} = 0$) requires a special investigation on which we shall not dwell.

The establishment of the converse relation causes no difficulty.

the analytic function $z_k(w)$; the function $z_k^{(s)}(w)$ is a holomorphic branch of that function.

Consequently, the point $w = b$ is said to be a *branch point* of the continuous mapping T^{-1} inverse to the holomorphic mapping T.

CASE 2. This holds when in any neighborhood of the point $z = a$ there is an infinite set of points which under the mapping τ go into the point $\underline{w} = \underline{b}$. In this case the point $z = a$ is said to be *exceptional* or *singular* for the holomorphic mapping τ. The mapping τ itself is then said to be *exceptional* or *singular* in the neighborhood of the point $z = a$. This point is also said to be *exceptional* or *singular* for the biholomorphic mapping T of the region D onto the region D^*.

As an example here[1] one may use the mapping $w_1 = z_1 z_2$, $w_2 = z_2$. All the points of the plane $z_2 = 0$ are carried by it into the origin of coordinates of the space of variables w_1, w_2, i.e., the point $w_1 = 0$, $w_2 = 0$. This mapping is exceptional in the neighborhood of all points of the plane $z_2 = 0$.

The mapping τ^{-1} inverse to this mapping has the form $z_1 = w_1/w_2$, $z_2 = w_2$. The point $w_1 = 0$, $w_2 = 0$ is an ambiguous point for the function $z_1(w)$. Evidently, whenever at least one of the functions $z_k(w)$ giving the mapping τ^{-1} has the point $\underline{w} = \underline{b}$ as an ambiguous point, the mapping τ has the point $z = a$ (where $\underline{b} = \tau a$) as an exceptional point.

The characteristic difference of the case under consideration from the preceding one consists in the fact that now one cannot continue the biholomorphic mapping T into the boundary point $z = a$ of the region D in such a way that the inverse mapping T^{-1} remains continuous at that point. This is easy to see from our example. The mapping τ^{-1} carries any neighborhood of the point $w_1 = 0$, $w_2 = 0$ into a region containing the entire plane $z_2 = 0$. At the point $w_1 = 0$, $w_2 = 0$, however, it has an irremovable discontinuity.

We note that when the complex dimension of a space is larger than two, then in the neighborhood of an exceptional point $z = a$ of the mapping τ and outside of the surfaces which are transformed along with the point $z = a$ into the point $w = b$, there can lie an infinite set of other exceptional points of that set. For example, for the mapping

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1) We note that the functions considered in the footnote on p. 107, also define an exceptional holomorphic mapping in the neighborhood of all points of the plane $1 + z_1 + z_2 = 0$.

$$w_1 = z_2 z_3, \quad w_2 = z_3 z_1, \quad w_3 = z_1 z_2 \tag{2.17}$$

the points (z_1, z_2, z_3) going into the point $w_1 = w_2 = w_3 = 0$ fill out the one-complex-dimensional analytic planes 1) $z_2 = z_3 = 0$, 2) $z_3 = z_1 = 0$, 3) $z_1 = z_2 = 0$. Nevertheless the exceptional points of the holomorphic mapping (2.17) fill out the two-complex-dimensional plane $z_1 = 0$, $z_2 = 0$, $z_3 = 0$.

REMARK. One should also remark that as a consequence of the Weierstrass preparation theorem, the points z at which the Jacobian $\partial w / \partial z$ is equal to zero are not isolated. All of them are preimages of the branch points of the mapping τ^{-1} inverse to the mapping τ, or are exceptional points of the mapping τ.

In conclusion, we consider the case when the Jacobian $\partial w / \partial z$ of the functions (2.14) is identically equal to zero in a region D. Then the mapping (2.14) puts the region D into correspondence with a certain set of surface elements of complex dimension less than n. The dimension of these elements, as follows from general theorems on implicit functions, depends on the rank of the matrix of the Jacobian in question.

In the case the mapping (2.14) is said to be *degenerate*.

3. Meromorphic mappings are generalizations of holomorphic mappings.

DEFINITION (meromorphic mapping in the narrow sense of a plane region over the space P^n). A continuous mapping T of the region D over the projective extended space P^n_z of variables $z_1, \cdots, z_n$ on the space D^* over the projective extended space P^m_w of variables $w_1, \cdots, w_m$ is said to be meromorphic (in the narrow sense) if: 1) it is holomorphic at all the points z of D to which there correspond finite points $Tz = w \in D^*$; 2) at the points z of D to which there correspond points at infinity $Tz = w \in D$, the mapping πT is holomorphic. Here π is the appropriate projective mapping (1.76) which carries the point $\underline{w} = \underline{Tz}$ into the finite point $\tilde{w}$.

The mapping T of the region D is said to be *meromorphic* (in the narrow sense) *at the point* z *of* D if there exists a subregion D_1 containing z in which that mapping is meromorphic.

It goes without saying that in the definition of a meromorphic mapping in the narrow sense one may restrict oneself to the consideration of a projective mapping π of the special kind.

REMARK. Analogously one defines a meromorphic mapping of a region over a function-theoretic space. In particular, we obtain such a mapping if in the

region D (which may be taken over complex projective space or over function-theoretic space), we define m meromorphic functions which do not have ambiguous points in that region.

If one rejects this last requirement one is led to nonunivalent mappings. Such mappings of a more general sort are also called meromorphic in a number of papers. We shall consider such mappings in the second, special volume of the present book. In what follows we consider only meromorphic mappings in the narrow sense, omitting the expression "in the narrow sense" whenever this will not lead to a misunderstanding.

We turn now to the case $m = n$. We agree to say that the mapping $T\{w = w(z)\}$ of the region D over the space P^n_z onto the region D^* over the space P^n_w is *generalized biholomorphic* or *bimeromorphic in the narrow sense* if both this mapping itself and its inverse T^{-1}, which maps the region D^* onto the region D, are meromorphic in the narrow sense. Analogously, one introduces the concept of generalized biholomorphic mapping at a point z of D.

Theorems 7.1, 7.3 and 10.1 easily extend to the case of meromorphic mappings. Suppose that in the region D over the space P^n_z one is given a system of n meromorphic functions $\tau\{w_k = w_k(z),\ k = 1, \cdots, n\}$. In order that the system τ should define a generalized biholomorphic mapping of the region D onto some region D^* over the space P^n_w it turns out to be sufficient that the Jacobian of this system should be different from zero in the region D. Here, if both the point z and the point $\underline{w} = \tau z$ are finite, then the Jacobian in question is constructed according to the usual rules. If the point z_0 is finite and the point $\underline{w}_0 = \tau z_0$ is at infinity, then in the neighborhood of the point z_0 the Jacobian is taken in the form $\partial(\tilde{w}_1, \cdots, \tilde{w}_n) / \partial(z_1, \cdots, z_n)$. Here at the points z of this neighborhood having finite images $\underline{w} = \tau z$,

$$\tilde{w}_k(z) = \frac{w_k(z)}{w_\nu(z)}, \quad \tilde{w}_\nu(z) = \frac{1}{w_\nu(z)};$$

$$k = 1, \cdots, \nu - 1, \ \nu + 1, \cdots, n. \tag{2.18}$$

At the points z of this neighborhood which have images $\underline{w} = \tau z$ at infinity, one takes an analytic continuation of the functions (2.18). The number ν is defined by the position of the point $\underline{w}_0$ at infinity (see in this connection subsection 3 of §5).

If conversely the point z_0 is at infinity and the point $\underline{w}_0 = \tau z_0$ is finite,

then we consider the Jacobian $\partial(w_1, \cdots, w_n)/\partial(\tilde{z}_1, \cdots, \tilde{z}_n)$, where

$$\tilde{z}_k = \frac{z_k}{z_\nu}, \quad \tilde{z}_\nu = \frac{1}{z_\nu}, \quad k = 1, \cdots, \nu - 1, \ \nu + 1, \cdots, n \tag{2.19}$$

(for an appropriate choice of the number ν). Finally, if both the points z_0 and $\underline{w}_0 = rz_0$ are at infinity, then we consider the Jacobian $\partial(\tilde{w}_1, \cdots, \tilde{w}_n)/\partial(\tilde{z}_1, \cdots, \tilde{z}_n)$, where the variables $\tilde{z}$ and $\tilde{w}$ are determined from relations (2.18) and (2.19).

The results of subsection 2 of this section all extend to the case of meromorphic mappings.

4. Complexly uniformizable boundary points of regions over the space P^n. Interiorly-branching regions.

DEFINITION (complexly uniformizable boundary points). The boundary point b of the region D over the projective extended space P^n of variables $z_1, \cdots, z_n$ is said to be complexly uniformizable if there exist functions $T\{z_s = \phi_s(t_1, \cdots, t_n)\}$ meromorphic in the polycylinder $U\{|t_k| < 1, \ k = 1, \cdots, n\}$ with center at the point $t = 0$, homeomorphically mapping that polycylinder on the extended neighborhood $\breve{V}_D(b)$ of the point b in the region D and carrying the point $t = 0$ into the point $z = b$.[1)]

It follows from Theorem 7.3 that if the Jacobian of the mapping T at the point $t = b$ is equal to zero, then the point b is a branch point for the region D. In this case the point b is said to be a *complexly uniformizable branch point*, and the variables $t_1, \cdots, t_n$ are *locally complex uniformizing parameters* in the neighborhood of the point b.

Above we have studied the properties of such a mapping, which carries the polycylinder U into the extended neighborhood of a branch point b of order m in the region D (see the discussion of Case 1 in subsection 2 of this section. There, to our region D corresponds the neighborhood $V_b(D_0^*)$, and to the variables t correspond the variables z). We saw there that this branch point belongs to the set $E^* = TE$, where E is a collection of analytic surfaces defined by the equation $\partial z/\partial t = 0$. Here $\partial z/\partial t$ is the Jacobian of the mapping T. This Jacobian is computed according to the usual rules if the point b is finite. If b is a point at infinity, then one must compute it according to the rules given in

1) For a more general definition of uniformizable point of the space, see the Introduction.

the preceding subsection of this section. All branch points of the region D lying in some sufficiently small extended neighborhood of the point b have one and the same branching order.

We observe that covering regions over the space P^n, including holomorphy and meromorphy regions, can have both complexly uniformizable and complexly non-uniformizable boundary points (see subsection 4 of §14 of Chapter III). Consequently, by *uniformizable singular points* of holomorphic and meromorphic functions we shall mean complexly uniformizable boundary points of their existence regions (of holomorphy and meromorphy). Their remaining singular points will be called *non-uniformizable.*

Observe that it will later turn out to be useful to extend the concept of covering region over the space P^n. Among the points of such a region one may also include the complexly uniformizable boundary points of the region. The set thus obtained is evidently a region, since: 1) extended neighborhoods of complexly uniformizable points are homeomorphic to the unit polycylinder, and 2) along with each such point there are also included the "adjacent" complexly uniformizable boundary points lying in the extended neighborhood of the point. If among the boundary points thus included there are also complexly uniformizable branch points, then the region obtained is said to be *interiorly-branching.* The branch points added in forming it are called interior branch points of this region or *interior critical points* of it. In what follows, if we do not specify the contrary, we restrict our consideration to regions over the space P^n without interior branch points.

In certain cases it turns out to be useful to adjoin to a covering region also its complexly non-uniformizable boundary points, satisfying specified conditions. As a result, we arrive at complex spaces which, in rather more general form, are considered in §§14–17 of the following chapter.

Observe that the concepts, "a region lies (strictly) inside another," "a region is a subregion of another," "a region covers another region" may be easily extended to the case of interiorly-branching regions.

We note that if the conditions of Osgood's Theorem 7.2 are satisfied for the mapping (2.14) for all points E of vanishing of the Jacobian $\partial w/\partial z$, then on the basis of the above remarks we may say that the functions (2.14) homeomorphically map the region D onto the interiorly-branching region D^*. In this case, as in the case considered in the preceding subsection, we call (2.14) a *generalized*

biholomorphic mapping of the region D onto the interiorly-branching region D^*.

This mapping is said to be uniformizable for the region D^*, and the variables $z_1, \cdots, z_n$ are uniformizing variables. Now we give the following definition.

DEFINITION (holomorphic and meromorphic functions in an interiorly-branching region). Functions defined in some interiorly-branching region over the space P^n are said to be holomorphic (meromorphic) in that region if they are holomorphic (meromorphic) at its noncritical points in the sense of the preceding definition and become holomorphic (meromorphic) at the critical points as a result of passing to locally complexly uniformizing parameters.

The definition of interiorly-branching regions of holomorphy or meromorphy does not differ from the definition of regions of holomorphy or meromorphy without interior critical points.

§11. PLANE REGIONS CONVEX RELATIVE TO SOME CLASS OF HOLOMORPHIC FUNCTIONS

1. Function classes. Let K be a set of functions meromorphic in some plane finite covering region D over the space P^n. This set K is said to be a *function class* if along with each function $f(z)$ it contains: 1) the derivatives of the function $f(z)$ of all orders; 2) all the functions of the type $A(f(z))^p$, where A is any complex number and p any positive integer.

In accordance with this definition of a function class, all the functions holomorphic in such a region, or all the functions meromorphic in a region, form a function class. Also the collection of all rational functions, all polynomials, all monomials of the form $az_1^{p_1}, \cdots, z_n^{p_n}$ (where $p_1, \cdots, p_n$ are integers) form function classes.

Another example of a class is that of a complete family of holomorphic functions which plays an important role in the theory of approximation. The set of functions $\mathfrak{F}$ holomorphic in some finite region D is said to be *a complete family of functions* if: 1) the family $\mathfrak{F}$ forms a ring of integrity in the algebraic sense; 2) for all $\nu = 1, \cdots, n$ the functions $f_\nu(z) \equiv z_\nu$ are in $\mathfrak{F}$; 3) all constant functions a are in $\mathfrak{F}$; 4) along with each function f, all the derivatives of the function f also belong to $\mathfrak{F}$.

Evidently a complete family of functions is formed by the set of functions holomorphic in some finite region D, or by the set of rational functions holomorphic

there, or the set of all polynomials.

In an infinite region D we consider only the class of all functions holomorphic in that region, or the class of all functions meromorphic there. The consideration of other function classes in an infinite region is made difficult by the fact that we have not introduced the concept of derivative of a holomorphic function at a point at infinity. One must also observe that the property of a set of functions of being a class is, generally speaking, not invariant under the transformations used in forming extensions of a space. It goes without saying that the classes of all holomorphic or all meromorphic functions in the region D have this invariance property.

NOTATIONS AND DEFINITIONS. 1) By the *distance* between two *finite* points z' and z'' in this section we shall understand the largest of the quantities $|z''_k - z'_k|$, $k = 1, \cdots, n$. In forming regions over the space C^n we shall make use of covers of the second kind. Any deviation from this rule will be specified.

2) If $f(z)$ is holomorphic or meromorphic in the region D, then we denote by $\sup|f(M)|$ the upper bound of the values of the modulus $|f(z)|$ at the points M of the region D.

3) Suppose that D is some region, M some subset of finite points of the region D, while $M \ll D$, and r is the lower bound of the limiting distances of the points M in the set D.

Suppose that we are given a number $q > 0$ with $q < r$. The collection of points of the region D which are distant from at least one point M of the subset by less than q forms an open subset of the region D which we shall denote by the symbol $M^{(q)}$. In addition we write $M^{(r)} = D$, $M^{(0)} = M$.

Now we suppose that K is some function class of holomorphic functions in the region D. Then the following theorem on simultaneous continuation of the functions of a class is valid.

THEOREM 11.1.[1] *If at the point* $P_0 \in D$ (where D is a finite region) *for each function* $f \in K$ *the following inequality holds:*

$$|f(P_0)| \leq \sup|f(M)|,$$

then: 1) *all the functions* $f \in K$ *are holomorphic in the polycylinder* $S(P_0, r)$; 2) *for all* $q < r$ *and all* $f \in K$,

1) See Cartan-Thullen [1].

$$\sup | f(S(P_0, q)) | \leq \sup | f(M^{(q)}) |.$$

PROOF. In order to simplify the notation we restrict our considerations to the case of two variables w, z. Suppose that $f(w, z)$ is some function belonging to the class K.

1) We consider the open set $M^{(q-\eta)} (\eta > 0)$ and write $\sup | f(M^{(q-\eta)}) | = A_f(\eta)$. Choose a point $Q \in M$. Then the bicylinder $S(Q, q - \eta)$ lies inside the set $M^{(q-\eta)}$, and therefore

$$\sup | f(S(Q, q - \eta)) | \leq A_f(\eta).$$

We write the Cauchy inequality (see subsection 4 of §3 in Chapter I) for the function $f(w, z)$:

$$\frac{1}{m!\, p!} \left| \frac{\partial^{m+p} f(Q)}{\partial w^m \partial z^p} \right| \leq \frac{A_f(\eta)}{(q - \eta)^{m+p}}. \tag{2.20}$$

By the hypotheses of our theorem and property 1) of the set of functions forming the class, inequality (2.20) will be satisfied also at the point P_0. With this in view, we consider the expansion of the function $f(w, z)$ into a double power series with center at the point $P_0(w_0, z_0)$:

$$f(w, z) = \sum_{m, p=0}^{\infty} a_{mp} (w - w_0)^m (z - z_0)^p. \tag{2.21}$$

In view of inequality (2.20) for the point P_0 we will have

$$| a_{mp} (w - w_0)^m (z - z_0)^p | \leq A_f(\eta) \frac{| w - w_0 |^m \, | z - z_0 |^p}{(q - \eta)^{m+p}}.$$

Thus, the modulus of the general term of the series (2.21) remains less than the general term of a convergent series so long as the point $P(w, z)$ lies in the bicylinder $S(P_0, \rho)$, where $\rho < q - \eta$. Therefore the convergence of the series (2.21) in that bicylinder follows. Now we let η go to zero. Then $\rho \longrightarrow q$, and thus the series (2.21) turns out to be convergent in the bicylinder $S(P_0, q)$. The number q may be taken as close as desired to r. Thus the first part of the assertion of our theorem is proved.

2) Now suppose that $q < q_1 - \eta$, where $q_1 < r$. Then for each function $f \in K$ we have in the bicylinder $S(P_0, q)$,

$$|f(w, z)| \leq \sum_{m, p=0}^{\infty} |a_{mp}(w - w_0)^m (z - z_0)^p| \leq$$

$$\leq \sum_{m, p=0}^{\infty} A_f(\eta) \frac{|w - w_0|^m |z - z_0|^p}{(q_1 - \eta)^{m+p}} \leq A_f(\eta) \sum_{m, p=0}^{\infty} \frac{q^{m+p}}{(q_1 - \eta)^{m+p}} =$$

$$= A_f(\eta) \frac{1}{\left[1 - \frac{q}{q_1 - \eta}\right]^2}. \tag{2.22}$$

Now we consider the ratio $(\sup |f(S(P_0, q))|)/A_f(\eta)$ for a fixed q and various η. If we show that this ratio is always less than or equal to unity, then this will prove the second assertion of our theorem, since then also in the limit as $\eta \longrightarrow q_1 - q$ the ratio will be ≤ 1, and we obtain (for $\eta = q_1 - q$)

$$\sup |f(S(P_0, q))| \leq A_f(q_1 - q) = \sup |f(M^{(q)})|.$$

Suppose that for some $\eta_0 > 0$ and $q < q_1 - \eta_0$ we have

$$\alpha = \frac{\sup |f(S(P_0, q))|}{A_f(\eta_0)} > 1.$$

Then we consider the function

$$\phi_p(w, z) = \left[\frac{f(w, z)}{A_f(\eta_0)}\right]^p$$

belonging to the class K (because of property 2) of the set of functions making up the class). Evidently $\sup |\phi_p(S(P_0, q))| = \alpha^p$. For sufficiently large p the quantity α^p is as large as desired. On the other hand,

$$A\phi_p(\eta_0) = \sup |\phi_p(M^{(q_1 - \eta_0)})| = \left[\frac{A_f(\eta_0)}{A_f(\eta_0)}\right]^p = 1.$$

Therefore in view of (2.22), since the function $\phi_p(w, z) \in K$,

$$\sup |\phi_p(S(P_0, q))| < \frac{1}{\left[1 - \frac{q}{q_1 - \eta_0}\right]^2}.$$

Thus, the functions $\phi_p(w, p)$ are uniformly bounded for all p in the bicylinder $S(P_0, q)$ and accordingly $\sup |\phi_p(S(P_0, q))|$ cannot be equal to α^p. We have

thus arrived at a contradiction, which forces us to drop the hypothesis made above. Thus our theorem is proved.

For the case of any region D over the space P^n we will consider only the class of all functions holomorphic in it. For this class of functions (and the region D, which now may be either finite or infinite), the following theorem is valid:

THEOREM 11.2. *If at the finite point $P_0 \in D$, for each function f, holomorphic in the region D, and some subset $M \ll D$ consisting of finite points, the inequality*

$$|f(P_0)| \leq \sup |f(M)|$$

is valid, then: 1) *all functions holomorphic in the region D are holomorphic in the polycylinder $S(P_0, r)$ and* 2) *for every $q < r$*

$$\sup |f(S(P_0, q))| \leq \sup |f(M^{(q)})|.$$

This theorem is proved quite analogously to the preceding.

2. K-convex regions. The theorems proved in the preceding subsection make it very useful to introduce the concept of convexity of a region with respect to some class of holomorphic functions. Such convexity turns out to be one of the most important properties of a holomorphy region.

DEFINITION (F-convex hull of a set in a region). Let F be some collection of functions holomorphic in the region D. By the F-convex hull $\hat{M}_F$ of a subset M of the region D we understand the set of all points z in D for which $|f(z)| \leq \sup |f(M)|$ for all functions $f \in F$.

If F is the collection of all functions holomorphic in the region D, then the set $\hat{M}_F$ is denoted by the symbol $\hat{M}$ and is said to be the holomorphically convex hull of the set M in the region D. Evidently the set $\tilde{M}_F$ is always closed in the region D and contains the set M.

We now apply the concept of the F-convex hull of a set to the case when the collection F is a class K of functions holomorphic in the region D.

DEFINITION (K-convexity, strong K-convexity, holomorphic convexity, strong holomorphic convexity). Suppose that K is some function class of functions holomorphic in the region D. The region D is called strongly K-convex if the following conditions are satisfied for it:

1) (the K-convexity condition). The K-convex hull of each compact subset

of the region D is compact (taking compactness with respect to the region D).

2) (the condition of K-separability). For each two distinct points z', $z'' \in D$ with the same coordinates there is a function $f \in K$ for which $f(z') \neq f(z'')$.

If the class K is the collection of all holomorphic functions in the region D, then these conditions are respectively called the conditions of holomorphic convexity of the region and of holomorphic separability. If they are satisfied the region D is said to be strongly holomorphically convex.

A holomorphically convex or strongly holomorphically convex region can be either finite or infinite. For other classes K the concept of K-convexity, and of K-strong convexity are only applied to finite regions. The property of a region being holomorphically or strongly holomorphically convex is conserved under all bimeromorphic (in the narrow sense) mappings, in particular for the mappings used in the construction of an extended space.

We note that in the case when K is the class of all functions holomorphic in the region D, the condition of holomorphic convexity may often be conveniently replaced by the following condition equivalent to it: *for each sequence of points $z^{(\nu)} \in D$ ($\nu = 1, 2, \cdots$) not having a limit point in the region D, one can find a function $f \in K$ holomorphic in the region D such that the sequence $|f(z^{(\nu)})|$ will not be bounded.*

We note further that in the case when the class K is the class of all functions holomorphic in the region D or when K is a complete family of holomorphic functions, one may, in the K-separability condition, take instead of the points z' and z'' with identical coordinates any pair of points of the region D, since for the points z', $z'' \in D$ with distinct projections the role of the function f may be played by one of its coordinates.

In 1953 K. Oka [1] proved the following important theorem.

THEOREM 11.3. *A holomorphically convex region always has the property of holomorphic separability.*

In other words: *a region is strongly holomorphically convex if it is holomorphically convex. The concepts of strong holomorphic convexity and of holomorphic convexity are equivalent to one another.*

Note that under certain additional hypotheses an analogous proposition holds also for regions holomorphic with respect to complete families of

holomorphic functions.[1)]

Evidently every finite region convex with respect to some class of holomorphic functions K_1 is holomorphically convex also with respect to the class of holomorphic functions K_2, where $K_1 \subset K_2$. Every finite region convex with respect to some class of holomorphic functions is holomorphically convex.

Now we give a precise formulation of the proposition mentioned at the beginning of the present subsection:

THEOREM 11.4. *If D is a holomorphy region for some function f_0 and $f_0 \in K$, where K is a class containing that function, then the region D is strongly K-convex.*

In particular: *every holomorphy region is holomorphically convex.*

PROOF. Suppose first that, in spite of the assertion of our theorem, the region D is not K-convex. Then there exists a compact subset M of the region D, whose K-convex hull $\hat{M}$ is not compact. From the hypothesis that the set M is compact it follows that there exists an open set $\mathfrak{M}$ such that $M \subset \mathfrak{M} \ll D$. We denote by $\mathfrak{M}^*$ the collection of finite points of the set $\mathfrak{M}$, and by r_0 the minimal limiting distance of the set $\mathfrak{M}$ (or, what is the same thing, of the set $\mathfrak{M}^*$) in the region D.

By hypothesis the K-convex hull $\hat{M}$ of the set M is not compact. Accordingly, there exists a sequence of points $P_k \in \hat{M}$ $(k = 1, 2, \cdots)$ converging in the region D. We select from the sequence of points $\underline{P}_k \in P^n$ a subsequence converging to some point $\underline{P} \in \underline{\partial D}$. Then we apply, if necessary, a transformation of the type (1.76) carrying the point $\underline{P}$ into the finite part of the space. As a result we obtain a sequence of points $P_k \in \hat{M}$ $(k = 1, 2, \cdots)$ for which $\lim_{k\to\infty} r_D(P_k) = 0$. On the other hand, from the fact that the point $P_k \in \hat{M}$ and from our hypothesis it follows that for all functions $f \in K$ we will have $|f(P_k)| \leq \sup|f(M)| \leq \sup|f(\mathfrak{M})| = \sup|f(\mathfrak{M}^*)|$ (the last inequality follows from the fact that $M \subset \mathfrak{M}$; this equality follows from the fact that in any neighborhood of a point at infinity of the set $\mathfrak{M}$ there is a finite point of that set). Then from Theorems 11.1 and 11.2 all the functions $f \in K$ are holomorphic in the polycylinders $S(P_k, r_0)$ and accordingly may be continued to the limits of the region D. In particular, it turns out to be possible to continue the function f_0 to the limits of the region D. This forces us to drop our supposition.

1) See Behnke-Stein [2].

Now suppose that the region D does not have the property of K-separability. Then there exist points P', $P'' \in D$ with identical coordinates such that the values of the function f_0 and of all its derivatives at these points coincide. As a consequence it is possible to continue the function f_0 to a region $\widetilde{D} > D$ (with $\underline{\widetilde{D}} = \underline{D}$) in which there corresponds to the points P' and P'' only one point $\widetilde{P}$ (this also holds for some sufficiently small neighborhoods of these points). Such a conclusion again contradicts the hypothesis of the theorem to be proved, and forces us to drop our second supposition. Thus Theorem 11.4 is completely proved. [1)]

REMARK. Recall (see subsection 4 of §10) that all our discussions relate to covering regions over the space P^n having no interior critical points. For covering regions having such points, Theorem 11.4 in general is not valid.

THEOREM 11.5. *The intersection of any finite or infinite collection of strongly K-convex regions is strongly K-convex.*

PROOF. Suppose that the region D is the intersection of the regions D_ν (where $\nu \in N$ and N is some set of indices). Then if $M \subset D$ is some compact subset of the regions in question, its K-convex hull $\widehat{M}$ in the region D is the intersection of its K-convex hulls $\widehat{M}_\nu$ in the regions D_ν. Here K is some class of holomorphic functions in the region D_ν. The intersection of the compact sets $\widehat{M}_\nu$ under our conditions turns out to be a compact set, so that the region D turns out to be K-convex.

That the condition of K-separability is satisfied for the region D is clear directly from the method of constructing the intersection region of a set of regions (see subsection 5 of §8).

From this theorem it follows that if for any boundary point Q of the region G over the space P^n one can find a function f_Q such that: 1) the holomorphy region of this region satisfies $D_{f_Q} > G$; and 2) the point Q is a singular point of the function f_Q (in other words, this point belongs to the boundary of the region D_{f_Q}), then the region G is a canonical covering for some holomorphy region.

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1) For a bounded and finite-sheeted region D the proof of Theorem 11.4 simplifies. In this case the K-convexity of the region D follows from the fact that from Theorem 11.1 $\widehat{M} \subset D_{r_0}$, where D_{r_0} is the collection of points of the region D with limiting distances larger than r_0.

Consider for example the hyperdisk $|w|^2 + |z|^2 < 1$. For each point (w_0, z_0) of the hypersphere $|w|^2 + |z|^2 = 1$ we choose a function $1/(ww_0 + zz_0 - 1)$. It has that point as a pole and is holomorphic in the hyperdisk $|w|^2 + |z|^2 < 1$. Accordingly this hyperdisk, which is a single-sheeted region, is a holomorphy region.

3. Sufficient conditions for a holomorphy region. In our later work the following theorem, a converse of Theorem 11.4, is of great significance.

THEOREM 11.6 (Cartan-Thullen [1]). *Suppose that K is some class of functions holomorphic in the region D. Then if that region is strongly K-convex, it is a holomorphy region.*

REMARK. If K is the class of all functions holomorphic in the region D, then the region D may be either finite or infinite. In other cases it is assumed that D is finite. The proof will be carried out here for the case of bounded regions.

PROOF. 1) We consider a canonical system of boundary points of the region D (see subsection 6 of §8) $M_1, M_2, \cdots$. We first construct a function holomorphic in the region D which has all the points M_i as essential singular points.

To this end we choose a countable sequence of points P_ν lying inside the region D with the following properties: suppose that the sequence $\{P_\nu\}$ has no limit points inside the region and each point M_i is a limit point for it. Then if ρ_ν is the limiting distance of the point P_ν in the region D it follows that $\lim \rho_\nu = 0$. [1)]

Further, we consider any sequence of regions D_ν $(\nu = 1, 2, \cdots)$ with the following properties:

a) each region D_ν is a subregion of the region D and $D_{\nu+1} \gg D_\nu$;

b) for each region D_0, where $D_0 \ll D$, one can find a number ν_0 such that if $\nu \geq \nu_0$ all the regions D_ν contain the region D_0 in the interior;

c) if r_ν is the minimal limiting distance of the region D_ν in the region D, then $\lim_{\nu\to\infty} r_\nu = 0$.

---

1) We use here the condition that the region is bounded. In the general case this conclusion in the metric employed may turn out to be invalid. It would be correct in the chordal metric.

Finally we require that the sequence of points P_ν in the regions D_ν be chosen so that $\rho_\nu < \hat{r}_\nu$. Here $\hat{r}_\nu$ is the minimal limiting distance of the K-convex hull $\hat{D}_\nu$ of the region D_ν in the region D. Then for each point P_ν there is in the class K a function $f_\nu(z)$ such that

$$|f_\nu(P_\nu)| > \sup |f_\nu(D_\nu)|. \tag{2.23}$$

Suppose that the condition $\rho < \hat{r}_\nu$ for the initial choice of the sequence of points P_ν and regions D_ν (satisfying the other requirements) is not satisfied. Then, using the fact that $\lim_{\nu\to\infty} P_\nu = 0$ and $\hat{D}_\nu \ll D$, can find for each m a quantity $N(m)$ such that for $\nu > N(m)$ we have $\rho_\nu < \hat{r}_m$. Then we select from the sequence $\{P_\nu\}$ a subsequence $\{P_{s_m}\}$ such that: 1) $s_m > N(m)$; 2) the distance of the point P_{s_m} to the point M_{t_m}, where $t_m = m - 2^{[\log_2 m]} + 1$ will be less than 2^{-m} (this is possible since all the points M_i are limit points for the points $\{P_\nu\}$). Here $[\log_2 m]$ is the integer part of $\log_2 m$. For $m = 1, 2, \cdots$ the numbers t_m exhaust the entire collection of natural numbers. Then we replace the sequence $\{P_\nu\}$ by the sequence $\{P_{s_m}\}$.

Now all the requirements are satisfied and relation (2.23) holds.

Without loss of generality we can put $f_\nu(P_\nu) = 1$ (dividing the functions by appropriate constants). This does not take us out of the limits of the class K. Then relations (2.23) are replaced by the inequalities

$$\sup |f_\nu(D_\nu)| < 1; \quad \nu = 1, 2, \cdots. \tag{2.24}$$

Now we define positive integers l_ν $(\nu = 1, 2, \cdots)$ such that

$$\sup |f_\nu(D_\nu)|^{l_\nu} < \frac{1}{\nu^2}. \tag{2.25}$$

Then we form the infinite product

$$\prod_{\nu=1}^{\infty} [1 - (f_\nu(z))^{l_\nu}].$$

In view of condition b) imposed on the choice of the regions D_ν and of inequalities (2.25), this infinite product converges uniformly in the region D. We put

$$f(z) = \prod_{\nu=1}^{\infty} [1 - (f_\nu(z))^{l_\nu}];$$

$f(z)$ thus turns out to be a holomorphic function in the region D.

It vanishes on every analytic surface $f_\nu(z) = 1$. Not more than a finite set of such surfaces can coincide with one another. In the contrary case there would exist an infinite sequence of values $\nu_1, \nu_2, \cdots$, such that the points P_{ν_1} would lie on all the surfaces $f_{\nu_k}(z) = 1$, i.e., for all values k we would have $f_{\nu_k}(P_{\nu_1}) = 1$. But one can always find such regions D_{ν_k} which contain the points P_{ν_1}. For these values of k the equation obtained would contradict inequalities (2.24).

Each point M_i of the canonical system of boundary points is a limit point of P_ν. Therefore each neighborhood of the point M_i will be cut by an infinite set of distinct null surfaces $f_\nu(z) = 1$. But in a sufficiently small neighborhood of the points where the function is holomorphic it is, because of the preparation theorem of Weierstrass, zero only on a finite set of analytic surfaces. Therefore all the points M_i are essential singular points of the function $f(z)$. Thus our assertion is proved.

2) It is not reasonable to expect that the function $f(z)$ constructed in the first part of our proof always has distinct functional elements at two distinct analytic points of the region D with equal coordinates (i.e., with identical projections). We shall now construct a function $f \cdot \phi$ (where $\phi(z)$ is holomorphic in the region D) which, like the function $f(z)$ itself, will have all the points M_i as essential singular points but will have distinct functional elements at all the points $M^{(s)}$ of the region D which lie over one geometrical point $\underline{M}$. Then our theorem will be completely proved.

What we shall do is the following: from all the pairs of elements S_i, S_j of a canonical covering of the region (see subsection 4 of §8) we select those for which $\epsilon_{ij} = 0$, although they also have analytic points with identical coordinates. We arrange these pairs in a sequence. We consider the lth pair of this sequence. Suppose that it consists of the elements S_i and S_j. In the elements S_i and S_j we choose two points M_l' and M_l'' with equal coordinates (but they must be distinct points of the region D).

Suppose that the function $f(z)$ constructed in the first part of our proof has equal functional elements at the points M_l' and M_l''. We select from the class K a function $\phi_l(z)$ having distinct values at the points M_l' and M_l'' (such a function necessarily exists in the class K because of the condition of K-separability). If conversely the function $f(z)$ has at the points M_l' and M_l''

distinct functional elements, then we put $\phi_l(z) \equiv 1$. Without loss of generality we may now suppose that the product $f\phi_l$ always has distinct values at the points M_l' and M_l'', i.e.,

$$f(M_l')\,\phi_i(M_l') \neq f(M_l'')\,\phi_l(M_l'')$$

(if necessary, we can arrange this by a small shift of the point M_l).

Now we select a sequence of positive numbers η_l such that the series $\Sigma_{l=1}^{\infty} \eta_l \sup|\phi_l(D_0)|$ converges for each subregion D_0 if $D_0 \ll D$. These numbers η_l are easily found by the diagonal process by considering any sequence of regions $D^{(s)} \ll D$, $s = 1, 2, \cdots$, exhausting the given region D.

Then we define positive numbers ρ_l, $\rho_l^{(k)}$ such that $0 < \rho_1 < \eta_1$, and we put

$$u_1 = \rho_1 |\phi_1(M_1')\,f(M_1') - \phi_1(M_1'')\,f(M_1'')| \neq 0. \tag{2.26}$$

Then we choose a sequence of numbers $0 < \rho_l^{(1)} < \eta_l$ for $l \geq 2$ such that

$$\sum_{l=2}^{\infty} \rho_l^{(1)} |\phi_l(M_l')\,f(M_l') - \phi_l(M_l'')\,f(M_l'')| < u_1, \tag{2.27}$$

and we put

$$\Phi(z) = \sum_{l=1}^{\infty} \rho_l \phi_l(z), \tag{2.28}$$

where $0 < \rho_l < \rho_l^{(1)}$ for $l = 2, 3, \cdots$, and $0 < \rho_1 < \eta_1$. Then the function $f\Phi$ at the points M_1' and M_1'' takes on distinct values, as one immediately sees from (2.27) and (2.26), and defines there distinct functional elements.

Now we choose numbers ρ_2, $\rho_l^{(2)}$ $(0 < \rho_2 < \rho_2^{(1)}$, $\rho_l^{(2)} < \rho_l^{(1)}$, $l = 3, 4, \cdots)$ such that for

$$u_2 = |\rho_1[\phi_1(M_1')\,f(M_1') - \phi_1(M_1'')\,f(M_1'')] +$$
$$+ \rho_2[\phi_2(M_2')\,f(M_2') - \phi_2(M_2'')\,f(M_2'')]|$$

the following inequality holds:

$$\sum_{l=3}^{\infty} \rho_l^{(2)} |\phi_l(M_2')\,f(M_2') - \phi_l(M_2'')\,f(M_2'')| < u_2.$$

If in (2.28) we choose $0 < \rho_1 < \eta_1$, $0 < \rho_2 < \rho_2^{(1)}$, $0 < \rho_l < \rho_l^{(2)}$ for $l = 3, 4, \cdots$, then the function $\Phi(z)$ defined there will be such that the product

$f\Phi$ takes on distinct values and defines distinct functional elements both at the points M_1' and M_1'' and at the points M_2' and M_2''.

Now it is evident how we must continue the process of choosing the coefficients ρ_l in order that the function $f\Phi$ should have distinct values, and should define distinct functional elements, at all the pairs of points M_i' and M_i'' in question.

On the other hand, the function $\Phi(z)$ is holomorphic in the region D. Therefore the product Φf has all the analytic surfaces $f_\nu = 1$ considered in the first part of the proof as null surfaces. Accordingly, all the points of the canonical system considered are essential singular points for the function Φf. Thus the region D turns out to be a holomorphy region for the function Φf. This completes the proof.

It follows from Theorem 11.5 that the intersection of a finite or infinite set of regions of holomorphy is a holomorphically convex region. Now it has been established that a holomorphically convex region is always a holomorphy region for some function. Thus we have proved the following theorem:

THEOREM 11.7. *The intersection of a finite or infinite set of holomorphy regions is always a holomorphy region for some function.*

The following theorem is also valid:

THEOREM 11.8. *Suppose that D is a holomorphy region over the space C_z^n of variables $z_1, \cdots, z_n$, and that E is a holomorphy region over the space C_w^m of variables $w_1, \cdots, w_m$. Then the region $D \times E$ is a holomorphy region over the space $C_z^n \times C_w^m$ of variables $z_1, \cdots, z_n, w_1, \cdots, w_m$.*

The proof of this theorem is evident.

As is known, for each disk $|z - a| < R$ one may find a function $f(z)$ holomorphic in this disk which has the circle $|z - a| = R$ as its natural boundary. All the points of this circle are singular points.

Therefore it follows from Theorem 11.8 that, for example, the polycylinder $\{|z_k - a_k| < R_k,\ k = 1, \cdots, n\}$ is a holomorphy region. An analogous conclusion is valid also for other polycylindrical regions.

4. Generalized theorem on the continuous distribution of singular points of a holomorphic function. Theorem 11.1 on the simultaneous continuation of a function class makes it possible to strengthen Theorem 9.1_1 in an essential way. This was the Hartogs theorem on the continuous distribution of singular

points. Thus the following theorem holds:

THEOREM 11.9 (Behnke-Sommer [1]). *Suppose that* $D \subset C^n$ *is a holomorphy region, and let the regions* $G_\mu \subset F_\mu$ $(\mu = 0, 1, 2, \cdots)$. *Here* F_μ *is a smooth surface of topological dimension* $k < 2n$ *on which, relative to holomorphic functions in the region* D *analytically continuable to some neighborhood of the region* G_μ, *the maximum principle* (see subsection 7 of §4) *holds, with*

$$\lim_{\mu\to\infty} F_\mu = F_0, \quad \lim_{\mu\to\infty} G_\mu = G_0, \quad \lim_{\mu\to\infty} \partial G_\mu = \partial G_0,$$

where the region G_0 *is bounded and* $\partial G_0 \subset D$. *Then, if the region* G_0 *contains a point not belonging to the region* D, *one can find a number* $M > 0$ *such that for* $\mu > M$ *each region* G_μ *will contain points not belonging to the region* D.

REMARK 1. Here a sequence of topologically k-dimensional surfaces $F_\mu \subset C^n$ $(\mu = 1, 2, \cdots)$ will be said to converge to the surface $F_0 \subset C^n$ if:

1) for each point $P_0 \in \overline{F}_0$ there is a sequence of points $P_\mu \in F_\mu$ $(\mu = 1, 2, \cdots)$ such that $\lim_{\mu\to\infty} P_\mu = P_0$ and conversely each point $P_0 = \lim_{\mu\to\infty} P_\mu$ (where $\mu = 1, 2, \cdots$, $P_\mu \in F_\mu$) either belongs to the surface F_0 or is a limit point of it;

2) for each closed region $\overline{G}_0 \subset F_0$ there exists a sequence of closed regions $\overline{G}_\mu \subset F_\mu$ $(\mu = 1, 2, \cdots)$ with the following property: for each number $\epsilon > 0$ there exists a number $M > 0$ such that for $\mu > M$

$$d(\overline{G}_\mu, \overline{G}_0) < \epsilon.$$

Here $d(\overline{G}_\mu, \overline{G}_0)$ is the upper limit of the distances of points of the closed region $\overline{G}_\mu$ from the closed region $\overline{G}_0$.

An analogous concept of convergence may be constructed for a sequence of sets $F_\mu \subset C^n$.

REMARK 2. As the surfaces F_μ in Theorem 11.9 one may take analytic surfaces consisting of ordinary points, since for these the maximum principle holds (see subsection 7 of §4). Then Theorem 11.9 easily reduces to Theorem 9.3. If we choose as the surfaces F_μ the analytic planes $\{z_j = a_j^{(\mu)}, j = 2, \cdots, n\}$ and as the regions G_μ the disks $\{|z_1 - a_1| < \epsilon\}$ in these planes, Theorem 11.9 easily reduces to Theorem 9.1_1.

PROOF OF THEOREM 11.9. Suppose that the region D is a holomorphy region for the function $f(z)$. Then by hypothesis this function is holomorphic at all the points of the boundary ∂G_0 and accordingly also in some neighborhood

$U(\partial G_0) = D_0$ of this boundary. We choose the region D_0 so that $\overline{D}_0 \subset D$, and suppose that the limiting distance is $r_D(D_0) = d$.

Consider the function class K: $\{g(z)\}$, which is formed in the region D by the function $f(z)$, its derivatives, polynomials in these function and the derivatives of these polynomials.

By the hypotheses of our theorem in the region G_0 there must be boundary points of the region D. Suppose that P_0 is one of them. Since the surface F_μ converges to the surface F_0 in the sense indicated above, we can find a number $M > 0$ such that for $\mu > M$,

$$d(\overline{G}_\mu, \overline{G}_0) < d, \ \ \partial G_\mu \subset D_0.$$

Suppose that there exists a number $\mu_0 > M$ such that the region $G_{\mu_0} \subset D$. Then all the functions of the class K are holomorphic in that region G_{μ_0}. Since for the surface F_{μ_0} the maximum principle holds, for all functions $g \in K$ and points $P \in G_{\mu_0}$ we have

$$|\, g(P) \,| \leq \sup |\, g(\partial G_{\mu_0}) \,| \leq \sup |\, g(D_0) \,|.$$

We have established that for the class K all the conditions of Theorem 11.1 are satisfied. Therefore all the functions of the class K, in particular the function $f(z)$, are holomorphic in all the polycylinders $S(P, d)$, where $P \in G_{\mu_0}$.

The point $P \in G_{\mu_0}$ may be chosen so that $S(P, d) \ni P_0$. This contradicts the hypotheses of Theorem 11.9, and our supposition is false.

This proves our theorem.

As Behnke and Sommer have made clear, if the maximum principle does not hold for surfaces F_μ, then the assertion of Theorem 11.9 is no longer valid.

The following strengthening of the theorem on the continuous distribution of singular points of a holomorphic function is due to Bremermann [3].

THEOREM 11.10. *Suppose that $D \subset C^n$ is a holomorphy region, and that $\{S_\mu\}$ and $\{T_\mu\}$ $(\mu = 0, 1, 2, \cdots)$ are two sequences of sets. If:*

1) $S_\mu \Subset D$, $T_\mu \Subset D$ *for* $\mu = 1, 2, \cdots$;

2) *for any function $f(z)$ holomorphic in the region D and any number μ the maximum modulus of the function $f|_{S_\mu \cup T_\mu}$ is attained on the set T_μ;*

3) *the limits $\lim_{\mu\to\infty} S_\mu = S_0$, $\lim_{\mu\to\infty} T_\mu = T_0$ exist, and the sets S_0 and T_0 are bounded;*

then if $T_0 \Subset D$, *also* $S_0 \Subset D$.

§12. ANALYTIC CLOSURE

1. Analytic closure in the sense of Hartogs.

DEFINITION. The region D of the space C^n of the complex variables $z_1, \cdots, z_n$ is said to be *analytically closed in the sense of Hartogs* at its boundary point a if:

1) whenever all the points of some neighborhood of the point a in the planes $z_j = a_j$, $j = 2, \cdots, n$, except the point a itself, lie in the region D, then for every $\epsilon > 0$ one can find a number $\delta > 0$ such that for every number b_1 satisfying the condition $0 < |b_1 - a_1| < \delta$ there corresponds a point $(b_1, b_2, \cdots \cdots, b_n)$, where $|b_j - a_j| < \epsilon$, $j = 2, \cdots, n$, which does not belong to the set D;

2) the first property is preserved under biholomorphic mappings of the neighborhood of the point a.

A region is said to be analytically convex in the sense of Hartogs if it is analytically convex in the sense of Hartogs at all of its finite boundary points. Evidently the boundary of a bounded region analytically convex in the sense of Hartogs is a perfect set. The intersection of regions analytically convex in the sense of Hartogs is a region analytically convex in the sense of Hartogs.

Theorems 9.1_1 of Hartogs and 9.2_1 of Levi establish that each holomorphy region or meromorphy region $D \subset C^n$ satisfies the first requirement of this definition at all of its boundary points. The second requirement is satisfied in view of the definition of biholomorphic mapping itself.

Thus we arrive at the following conclusion:

A holomorphy region or meromorphy region $D \subset C^n$ *is always analytically convex in the sense of Hartogs at all of its boundary points.*

In the definition of analytic convexity in the sense of Hartogs one may, in using the second part of the definition, replace the family of analytic planes $z_j = b_j$, $j = 2, \cdots, n$, by any regular family of one-complex-dimensional analytic surfaces (see subsection 6 of §9).

Repeating the argument of Theorem 9.3, we arrive at the following proposition.

THEOREM 12.1. *For the region* $D \subset C^n$ *to be analytically convex in the sense of Hartogs at its boundary point* $a \in C^n$, *it is necessary and sufficient*

that every family $\{E_\alpha\}$ *of one-complex-dimensional analytic surfaces, regular in a neighborhood* U_a *of that point* a, *should possess the following property:*

If the point $a \in E_{\alpha 0}$ *and* $[(E_{\alpha 0} \cap U_a) - a] \subset D$, *then there exists a neighborhood* $V_{\alpha 0}$ *of the point* α^0 *such that for each* $\alpha \in V_{\alpha 0}$ *the set* $U_a \cap E_\alpha$ *contains points not belonging to the region* D.

We note some properties of regions analytically convex in the sense of Hartogs.

THEOREM 12.2. *Suppose that* P *is a point of some hypersphere* E. *If the part of a neighborhood of the point* P *lying outside the hypersphere* E *belongs to the region* $D \subset C^n$ *analytically convex in the sense of Hartogs, then so does the point* $P \in D$ *itself.*

PROOF. We take P to be the origin of coordinates. Then with proper orientation the center of the sphere will be at a point $(-R, 0, \cdots, 0)$, and the planes $z_j = 0$, $j = 2, \cdots, n$, touch the hypersphere E at the point P and lie outside it.

Suppose that contrary to our assertion $P \in \partial D$. Then, in view of the definition of a region analytically convex in the sense of Hartogs, to each number $z_1 = \eta$, where $0 < \eta < \delta$, there corresponds a point $(\eta, z_2, \cdots, z_n) \notin D$, where all the $|z_j| < \epsilon$ $(j = 2, \cdots, n)$. The numbers ϵ and δ may always be chosen so that the point $(\eta, z_2, \cdots, z_n)$ will belong to the part of a neighborhood of the point P mentioned in the conditions of Theorem 12.2. We arrive at a contradiction and must drop the above supposition as invalid. Theorem 12.2 is proved.

THEOREM 12.3. *Suppose that* $\mathfrak{S}$ *is a perfect set of points, and* Q *any fixed point of the space* C^n. *Suppose that there exists a point* $P_0 \in \mathfrak{S}$ *such that* $d(Q, P_0) \geq d(Q, P)$ *for all points* $P \in \mathfrak{S}$ *and lying in some neighborhood of the point* P_0. *Here* d *is the distance between corresponding points. Then there does not exist a region* $D \subset C^n$ *analytically convex in the sense of Hartogs with the following properties:*

1) $\mathfrak{S} \subset \partial D$; *in some neighborhood of the point* P_0 *the boundary of the region* D *consists entirely of points of the set* $\mathfrak{S}$.

2) *In the neighborhood of the point* P_0 *the region* D *intersects only with the extension of the segment* QP_0.

PROOF. We take the point P_0 as the point P of the preceding theorem, and the point Q as the center of the hypersphere E. If there did exist a region

D with the properties indicated in Theorem 12.3, then to it (within the limits of some neighborhood of the point P_0) must belong the exterior of the hypersphere E. Therefore, from Theorem 12.2, our assertion results.

From the theorems just proved we also obtain certain corollaries:

THEOREM 12.4. *The boundary of the region $D \subset C^n$, analytically convex in the sense of Hartogs, cannot have a bounded perfect part other than the entire boundary itself.*

REMARK. Speaking about an isolated part T of the boundary ∂D, we wish to say here that if a point $P \in T$, then all the points of the boundary ∂D which lie in some neighborhood of the point P also belong to that part T.

THEOREM 12.5. *If the boundary ∂E of some bounded region $E \subset C^n$ lies in the region $D \subset C^n$, analytically convex in the sense of Hartogs, then $E \subset D$.*

It follows from this last theorem that if the function $f(z)$ is holomorphic (meromorphic) at all the points of the boundary ∂E of some bounded region $E \subset C^n$ and remains single-valued for all of its analytic (meromorphic) continuations into the region E, then it can be analytically (meromorphically) continued to the entire region E.

We note that the requirement of single-valuedness of the continuations of the function $f(z)$, although essential for this method of proving the theorem, is in fact superfluous (see Theorem 21.2 in Chapter IV).

2. Analytic convexity in the sense of Levi. We now will restrict ourselves to the consideration of holomorphy regions and meromorphy regions in the space C^2 of variables $w = u + iv$, $z = x + iy$ (the general case will be considered in the second part of this book). In the case when the boundary of such a region is a hypersurface of class $\mathcal{C}^2$, the condition that it be analytically convex in the sense of Hartogs may be replaced by the condition that it be analytically convex in the sense of E. E. Levi. This condition turns out to be geometrically more intuitive and analytically easier to verify.

DEFINITION (analytic convexity in the sense of E. E. Levi). The hypersurface $\Phi(u, v, x, y) = \Phi(w, z) = 0$ (or, what is the same thing, the region adjoining that hypersurface on the side $\Phi < 0$) is said to be analytically convex in the sense of Levi at the ordinary point P if, within the limits of some neighborhood of the point P, on all analytic surfaces passing through this point there exist points other than P at which $\Phi \geq 0$.

Note that this definition is completely analogous to the usual definition of

convexity of a curve on a plane at a point P of that plane. The only difference is that instead of the set of analytic surfaces passing through the point P one takes the bundle of straight lines with vertex at the point P.

REMARK. In the sequel, if the hypersurface $\Phi = 0$ belongs to the boundary ∂D of the region D, then, unless specified otherwise, we will suppose that the region D lies on the side $\Phi < 0$.

If the hypersurface $\{\Phi = 0\} \subset \partial D$, where the region $D \subset C^2$ is analytically convex in the sense of Levi, then that region is analytically convex in the sense of Hartogs at all the points $P \in \{\Phi = 0\}$. This follows immediately from Theorem 12.1. The converse proposition also holds. We shall prove it for hypersurfaces $\Phi = 0$ of class $\mathcal{C}^2$.

THEOREM 12.6. *If the hypersurface $\Phi = 0$ of class $\mathcal{C}^2$ in the neighborhood of one of its ordinary points is the boundary of some region $D \subset C^2$, analytically convex at the point P in the sense of Hartogs, then that hypersurface is analytically convex in the sense of Levi at the point P.*

PROOF. PART 1. First of all we observe that if (ω, ζ) is an ordinary point of some analytic surface $\phi(w, z) = 0$, then one may construct in the following way a family of analytic surfaces regular in the neighborhood of the point (ω, ζ) and including the surface $\phi(w, z) = 0$. We will shift all the points of the surface $\phi(w, z) = 0$ parallel to one another, defining the shift of the points of that surface by vectors constant along the surface (each surface of the family is defined by the vector which determines the shift) and parallel to some plane $w - \omega = p(z - \zeta)$. Here p is chosen so that this plane is not tangent to the surface $\phi(w, z) = 0$ at the point (ω, ζ) (since (ω, ζ) is an ordinary point of the surface, this may always be done by avoiding some definite value of p). It is easily seen that the equation of the family of surfaces thus constructed is written in the form

$$\Psi(w, z, \alpha) = \phi(w + p\alpha, z + \alpha) = 0.$$

All three conditions of regularity of the family in our case are evidently satisfied: 1) for $\alpha = 0$, $\Psi(\omega, \zeta, 0) = \phi(\omega, \zeta) = 0$; 2) $(\partial\Psi/\partial\alpha)_{\omega,\zeta,0} = p(\partial\phi/\partial w)_{\omega,\zeta,0} + (\partial\phi/\partial z)_{\omega,\zeta} \neq 0$, since the plane $w - \omega = p(z - \zeta)$ is not tangent to the surface $\phi(w, z) = 0$ at the point (ω, ζ); 3) by hypothesis for the surface $\phi(w, z) = 0$ the point (ω, ζ) is ordinary; we may assume that $(\partial\phi/\partial w)_{\omega,\zeta} \neq 0$. Therefore it follows that $(\partial\Psi/\partial w)_{\omega,\zeta,0} = (\partial\phi/\partial w)_{\omega,\zeta} \neq 0$. Thus the family of analytic surfaces $\Psi = 0$ is regular in the neighborhood of the point (ω, ζ).

PART 2. Now we turn to the direct verification of the assertion of the theorem. Suppose that our assertion were not true and that the hypersurface were not analytically convex in the sense of Levi. Then there would exist an analytic surface passing through the point P and lying, except for that point, within the limits of a neighborhood of it and in the part of the space where $\Phi < 0$. The equation of this surface may always be represented in the form $z = \chi(w)$ (see formula (2.33_1) in the proof of the following theorem).

Further we construct a regular family of analytic surfaces by shifting the surface $z = \chi(w)$ (as indicated in the first part of the proof). We may do this, since the equation of the original surface $z = \chi(w)$ is solvable with respect to one of the variables in the neighborhood of the point P, which is consequently an ordinary point of the surface. The shifts used in the construction of the family of surfaces are taken to be parallel to an analytic plane passing through the normal to the hypersurface. It is geometrically evident that among the analytic surfaces of the family thus obtained there will be some which, within the limits of some neighborhood of the point P, lie entirely in the part of the space where $\Phi < 0$ (in any case, this can be verified analytically by using Taylor's theorem). This contradicts Theorem 12.1. Thus the supposition is false and the theorem is proved.

REMARK. At a point P of the hypersurface which is a natural boundary for a function $f(z)$, this function cannot be meromorphic. In this case the set of singular points of the function $f(z)$ in the neighborhood of the point P is not sparse. Thus the hypersurface consists of essential singular points of the function and is a natural boundary for its meromorphy.

3. **Levi's condition.** E. E. Levi gave a simple criterion for analytic convexity of a hypersurface.[1] This criterion is expressed by the following theorem.

THEOREM 12.7. *In order that the hypersurface* $\Phi = 0$ (*the function* Φ *belongs to the class* $\mathcal{C}^2$) *should be analytically convex in the sense of Levi at an ordinary point* P, *it is necessary that at that point*

$$L(\Phi) = -\begin{vmatrix} 0 & \Phi'_{\bar{w}} & \Phi'_{\bar{z}} \\ \Phi'_w & \Phi''_{w\bar{w}} & \Phi''_{w\bar{z}} \\ \Phi'_z & \Phi''_{z\bar{w}} & \Phi''_{z\bar{z}} \end{vmatrix} \geq 0, \tag{2.29}$$

1) See Levi [1, 2].

and sufficient that there $L(\Phi) > 0$.

The quantity $L(\Phi)$ is usually called *Levi's determinant.*

REMARK. Therefore it follows that in order that meromorphy regions should adjoin the hypersurface $\Phi = 0$ on both sides it is necessary that the condition $L(\Phi) = 0$ be satisfied.

PROOF. We take P to be the origin of coordinates and represent the equation of the hyperplane tangent to the surface $\Phi = 0$ at the point P in the form

$$(\Phi'_w)_0\, w + (\Phi'_z)_0\, z + (\Phi'_{\overline{w}})_0\, \overline{w} + (\Phi'_{\overline{z}})_0\, \overline{z} =$$

$$= 2\,\mathrm{Re}\,[(\Phi'_w)_0\, w + (\Phi'_z)_0\, z] = 0. \tag{2.30}$$

Here $(\Phi'_w)_0\, w + (\Phi'_z)_0\, z = 0$ is the equation of the analytic plane passing through the point P and lying in the hyperplane (2.30). Such an analytic plane is unique: it touches the hypersurface $\Phi = 0$ at the point P. Then we carry out the mapping

$$w_1 = \alpha w + \beta z, \quad z_1 = \left(\frac{\partial \Phi}{\partial w}\right)_0 w + \left(\frac{\partial \Phi}{\partial z}\right)_0 z. \tag{2.31}$$

Here the quantities α, β are chosen so that $|w_1|^2 + |z_1|^2 = |w|^2 + |z|^2$ (then the mapping (2.31) will be a rotation). With such a choice of coefficients the mapping (2.31) may be inverted. Under the mapping (2.31) and the mapping inverse to it, analytic surfaces go again into analytic surfaces, and the relations of intersection also do not change. Therefore, if the hypersurface obtained as a result of the mapping is analytically convex, then also the initial hypersurface must have been analytically convex, and on the same side. Carrying out the mapping (2.31), we return to the old notation for the variables.

After this transformation the left side of the equation of the hypersurface under investigation may be represented as follows:

$$\Phi = \frac{z + \overline{z}}{2} + aw^2 + \overline{aw}^2 + bw\overline{w} +$$

$$+ [\text{second-order terms containing } z] +$$

$$+ \eta(|w|^2 + |z|^2). \tag{2.32}$$

Here $\lim_{|w|^2 + |z|^2 \to 0} \eta = 0$. Since Φ is a real function, the coefficients on w^2, $\overline{w}^2$ are conjugate, and b is real.

Because of the definition of analytic convexity, only those analytic surfaces

are essentially significant for us which, in the neighborhood of P, pass along only one side of the hypersurface.

Suppose that $f(w, z) = 0$ is the equation of an analytic surface passing through the point P. Then, by Theorem 4.12, the equation $f(w, z) = 0$ may be replaced in the neighborhood of P by the equation $w = 0$ or by the equation

$$z = A_1 w^{1/m} + A_2 w^{2/m} + \cdots + A_m w + B_1 w^{1+\frac{1}{m}} + \cdots + B_m w^2 + \cdots. \tag{2.33}$$

Substituting z from (2.33) into (2.32), we see that if at least one of the coefficients $A_1, \cdots, A_m, B_1, \cdots, B_{m-1}$ is different from zero, then the sign of the function Φ at the points of the surface $f = 0$ close to the origin P (determined by the sign of the term having the lowest order of smallness) will coincide with the sign of the quantity

$$\frac{\overline{z} + z}{2} = \frac{cw^{k/m} + \overline{cw}^{k/m}}{2} = \operatorname{Re}(cw^{k/m}); \quad \frac{k}{m} < 2$$

(here c is the coefficient of the lowest non-vanishing term in (2.33)). One can always prescribe the argument w so that this expression will have a sign given in advance. Therefore, in the neighborhood of the point P, this analytic surface passes on both sides of the hypersurface.

For the analytic surface $w = 0$ the lowest term in the expression (2.33) has the form $(z + \overline{z})/2$. In the neighborhood of the origin this surface also passes along both sides of the hypersurface. Now it remains to consider analytic surfaces for which the expansion (2.33) has the form (we write $B_m = 2B$)

$$z = 2Bw^2 + \cdots. \tag{2.33$_1$}$$

For the points of such an analytic surface

$$\Phi = (B + a)\, w^2 + (B + \overline{a})\, \overline{w}^2 + bw\overline{w} + \cdots =$$

$$= 2\operatorname{Re}\,[(B + a)\, w^2] + bw\overline{w} + \cdots. \tag{2.34}$$

The remaining terms, denoted by the dots, have a higher order of smallness. For each B and a the argument w may be chosen so that $\operatorname{Re}\,[(B + a)\, w^2] = 0$.

Therefore, if $b > 0$, then each analytic surface passing through the point P has points in that part of the neighborhood of the point P where $\Phi > 0$ (if $b < 0$, then in the part where $\Phi < 0$). Further, if $b > 0$ (respectively, $b < 0$), then there exists an analytic surface, in fact $z = -2aw^2$, which within the

limits of some neighborhood of the point P lies entirely in that part of that neighborhood where $\Phi > 0$ (or respectively $\Phi < 0$). This follows from the fact that at the points of that surface, from (2.34), $\Phi = bw\overline{w} + \cdots$. Thus we arrive at the conclusion:

1. If $b > 0$, then the hypersurface is analytically convex at the point P.

2. In the case $b = 0$ the question as to the direction of analytic convexity of the hypersurface remains open (on this subject see subsection 3 of the present section).

The coefficient $b = 4[\Delta_w \Phi(w, \overline{w}, 0, 0)]_P$. Here the Laplace operator Δ_w, acting on the function Φ, is computed in the plane $z = 0$, i.e., in the analytic plane that is tangent to the hypersurface. In the general case this plane will have the equation $w\Phi'_w + z\Phi'_z = 0$, and on it we will have $w = \Phi'_z t$, $z = -\Phi'_w t$. Here and in the remainder of this subsection all the derivatives of Φ are taken at the point P, with t a complex parameter. By calculation we find that

$$\Delta_t \Phi(\Phi'_z t,\ \overline{\Phi'_z t},\ -\Phi'_w t,\ -\Phi'_{\overline{w}} \overline{t}) = L(\Phi).$$

It is easy to see that if the equation $\Phi = 0$ has the form (2.32), then $L(\Phi) = b/4$. We must now verify the fact that $L(\Phi)$ preserves its sign under a biholomorphic mapping of a region of the space in which the piece of the hypersurface lies, the mapping being defined by means of holomorphic functions of the form

$$w_1 = w_1(w, z); \quad z_1 = z_1(w, z). \tag{2.35}$$

If $\Phi_1 = \Phi[w_1(w, z),\ z_1(w, z)]$,

$$\left.\begin{aligned}
&\frac{\partial \Phi_1}{\partial w_1} = \Phi'_w \frac{\partial w}{\partial w_1} + \Phi'_z \frac{\partial z}{\partial w_1}; \quad \frac{\partial \Phi_1}{\partial z_1} = \Phi'_w \frac{\partial w}{\partial z} + \Phi'_z \frac{\partial z}{\partial z_1};\\
&\frac{\partial^2 \Phi_1}{\partial \overline{w}_1 \partial \overline{w}_1} = \Phi''_{w\overline{w}} \frac{\partial w}{\partial w_1}\frac{\partial \overline{w}}{\partial \overline{w}_1} + \Phi''_{w\overline{z}} \frac{\partial w}{\partial w_1}\frac{\partial \overline{z}}{\partial \overline{z}_1} +\\
&\qquad\qquad + \Phi''_{z\overline{w}} \frac{\partial z}{\partial z_1}\frac{\partial \overline{w}}{\partial \overline{w}_1} + \Phi''_{z\overline{z}} \frac{\partial z}{\partial z_1}\frac{\partial \overline{z}}{\partial \overline{z}_1}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{2.36}$$

Putting $L(\Phi_1)$ into the new variables, we find as the result of calculation that

$$L(\Phi_1) = L(\Phi)\left|\frac{\partial(w, z)}{\partial(w_1, z_1)}\right|^2. \tag{2.37}$$

Thus the sign of the expression $L(\Phi)$ does not change under the mapping (2.35). On the other hand, the mapping (2.31) is a special case of the mapping (2.35).

Thus our theorem is proved.

REMARK 1. Suppose that the system of coordinates is chosen so that the equation $\Phi = 0$ has the form (2.32). Then if $b > 0$ it turns out that all the surfaces $z + \alpha w^2 = 0$, where α is a complex parameter with $|\alpha - 2a| < b$, pass in the neighborhood of the point P on the side $\Phi > 0$. The radius of this disk b, which is so to speak a normalized value of the Levi determinant, will in what follows be called the *degree* of analytic convexity of the hypersurface at the point P. We leave to the reader the verification of the properties of the hypersurface indicated here.

REMARK 2. It follows from equation (2.37) that the Levi determinant is preserved under biholomorphic mappings.

Theorem 12.7 may be essentially strengthened. The following theorem holds

THEOREM 12.7_1. *In order that the hypersurface* $\Phi = 0$, *belonging to the class* $\mathcal{C}^2$, *should be analytically convex in the sense of Levi, it is necessary and sufficient that* $L(\Phi) \geq 0$.[1]

4. Analytic hypersurfaces. We turn to the study of hypersurfaces at all of whose points we have $L(\Phi) = 0$ (we shall leave aside the case that $L(\Phi) = 0$ at isolated points and on parts of $\Phi = 0$ of a smaller number of dimensions). Such a hypersurface may, it turns out, be made up of analytic surfaces, and is the so-called "analytic hypersurface."

DEFINITION (element of an analytic hypersurface). An element of a hypersurface lying in some region D is said to be *analytic* if it can be given by the equation

$$h(w, z, t) = 0 \quad (a < t < b). \tag{2.38}$$

Here t is a real parameter on which the function h depends continuously; for each $t_0 (a < t_0 < b)$ $h(w, z, t_0)$ is a holomorphic function of w, z at every point $P(w_0, z_0) \in D$, where $h(w_0, z_0, t_0) = 0$. The equation $h(w, z, t_0) = 0$ must define an analytic surface element containing the point P.

Therefore an analytic hypersurface of the indicated type may always be decomposed into analytic surfaces. Furthermore, we shall prove a theorem of

1) For the proof of this proposition see for example Chapter III of the paper of Fuks [2] or the second part of the present book (to be published).

the following type, due to E. Levi.

THEOREM 12.8. *Given a hypersurface of class* $\mathcal{C}^2$, *consisting of ordinary points*

$$\Phi(u, v, x, y) \equiv \Phi(w, z) = 0.$$

1. *If this hypersurface is analytic, then at all of its points* $L(\Phi) = 0$.

2. *If at all of its points* $L(\Phi) = 0$, *then in the neighborhood of each of its points this hypersurface is analytic.*

PROOF. The first assertion of the theorem, expressing the necessity of the condition $L(\Phi) = 0$ for an analytic hypersurface, is almost evident. If P is a point of that hypersurface, then through it there passes some analytic surface $h(w, z, t_0) = 0$ consisting entirely of points of the hypersurface. We represent the equation $\Phi = 0$ in the form (2.32) and suppose that P is an ordinary point[1] of the surface $h = 0$. Then this surface can be given in the neighborhood of P by the equation $z = Aw + Bw^2 + \cdots = \phi(w)$. We have $\Phi(w, \phi(w)) \equiv 0$. Hence $A = 0$, $B = -a$, $b = 0$. Accordingly, $L(\Phi) = 0$, as was required to be proved.

Turning to the proof of the second assertion we consider the hypersurface $\Phi = 0$ in the neighborhood of one of its points P. The point P is an ordinary point for it, i.e., not all the partial derivatives of the function Φ at that point are equal to zero. We suppose that at the point P,

$$\frac{\partial \Phi}{\partial u} - i\frac{\partial \Phi}{\partial v} = \frac{\partial \Phi}{\partial w} \neq 0.$$

We set as our goal to show that when $L(\Phi) = 0$, there exists in the neighborhood of P a family of analytic surfaces $w = w(z, t)$, exhausting all the points of our hypersurface in this neighborhood and consisting of points of it. To this end we will seek in some neighborhood U_P of the point P analytic surfaces

$$w = w(z), \text{ or } u = u(x, y), \quad v = v(x, y), \tag{2.39}$$

lying entirely on the hypersurface $\Phi = 0$.

If the surface (2.39) in this neighborhood entirely lies on the hypersurface $\Phi = 0$, then the following will hold there identically:

$$\Phi(u(x, y), v(x, y), x, y) \equiv 0. \tag{2.40}$$

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1) If the point P is not ordinary, it may be considered as limiting with respect to some set of ordinary points. In view of the continuity of the partial derivatives with respect to Φ we will have $L(\Phi) = 0$ at that point also.

Differentiating this, we obtain

$$\left.\begin{aligned}\Phi'_x + \Phi'_u \frac{\partial u}{\partial x} + \Phi'_v \frac{\partial v}{\partial x} = 0,\\ \Phi'_y + \Phi'_u \frac{\partial u}{\partial y} + \Phi'_v \frac{\partial v}{\partial y} = 0.\end{aligned}\right\} \tag{2.41}$$

The functions $u(x, y)$, $v(x, y)$ must moreover satisfy the Cauchy-Riemann conditions, i.e., we must have $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$. Therefore in order to determine the function $u(x, y)$ we obtain the following system of differential equations:

$$\left.\begin{aligned}\Phi'_x + \Phi'_u \frac{\partial u}{\partial x} - \Phi'_v \frac{\partial u}{\partial y} = 0,\\ \Phi'_y + \Phi'_v \frac{\partial u}{\partial x} + \Phi'_u \frac{\partial u}{\partial y} = 0.\end{aligned}\right\} \tag{2.42}$$

We shall now show that since the quantities x, y, u, v are connected by the equations $\Phi = 0$ and $L(\Phi) = 0$, we get:

a) the system of equations (2.42) is complete, i.e., for the quantities $\partial u/\partial x$ and $\partial u/\partial y$ determined by it we have the equation

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right);$$

b) the function $u(x, y)$ determined by (2.42) is harmonic, i.e.,

$$\Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 0.$$

From (2.42) it follows that

$$\left.\begin{aligned}\frac{\partial u}{\partial x} = A = \frac{-\Phi'_u\Phi'_x - \Phi'_v\Phi'_y}{\Phi'^2_u + \Phi'^2_v};\\ \frac{\partial u}{\partial y} = B = \frac{-\Phi'_u\Phi'_y + \Phi'_v\Phi'_x}{\Phi'^2_u + \Phi'^2_v}.\end{aligned}\right\} \tag{2.43}$$

We wish to show that in our case $\partial A/\partial y - \partial B/\partial x = \partial A/\partial x + \partial B/\partial y = 0$. To this end we use the following equation, which results from the definition of the formal derivative (equation (1.4)):

$$\left\{\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right\}+i\left\{\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}\right\}=\frac{\partial(A-Bi)}{\partial x}+i\frac{\partial(A-Bi)}{\partial y}=2\frac{\partial(A-Bi)}{\partial\overline{z}}. \tag{2.44}$$

On the other hand, in formal derivatives,

$$A-Bi=-\frac{(\Phi_u'\Phi_x'+\Phi_v'\Phi_y')-i(\Phi_u'\Phi_y'-\Phi_v'\Phi_x')}{\Phi_u'^2+\Phi_v'^2}=$$

$$=-\frac{\Phi_x'-i\Phi_y'}{\Phi_u'-i\Phi_v'}=-\frac{\Phi_z'}{\Phi_w'}. \tag{2.45}$$

Thus the condition

$$\frac{\partial}{\partial\overline{z}}\left[\frac{\Phi_z'}{\Phi_w'}\right]=0 \tag{2.46}$$

is equivalent to both the requirements a) and b). In calculating the expression (2.46) we must take $\partial u/\partial x=A$, $\partial u/\partial y=B$, and calculate $\partial v/\partial x$ and $\partial v/\partial y$ (starting from the fact that x, y, u, v are connected by the relation $\Phi=0$) from equation (2.41).[1] Putting $\partial u/\partial x=A$, $\partial u/\partial y=B$ and using (2.43) we find that

$$\frac{\partial v}{\partial x}=-B,\quad \frac{\partial v}{\partial y}=A. \tag{2.47}$$

This is equivalent to the equations (used below in transforming equation (2.46))

$$\frac{1}{2}\left[\frac{\partial(u+iv)}{\partial x}+\frac{\partial(u+iv)}{\partial y}\right]=\frac{\partial w}{\partial\overline{z}}=0;$$

$$\frac{\partial\overline{w}}{\partial z}=\frac{1}{2}\left[\frac{\partial(u-iv)}{\partial x}+i\frac{\partial(u-iv)}{\partial y}\right]=-\frac{\Phi_x'-\Phi_y'}{\Phi_u'+i\Phi_v'}=-\frac{\Phi_{\overline{z}}'}{\Phi_{\overline{w}}'}. \tag{2.48}$$

Now we obtain

1) At the point P we have $\Phi_w'\neq 0$ and consequently either $\Phi_u'\neq 0$ or $\Phi_v'\neq 0$. Suppose that $\Phi_v'\neq 0$. If at the point P we have $\Phi_v'=0$, then there $\Phi_u'\neq 0$, and we may interchange the rôles of u and v.

$$\frac{\partial}{\partial \bar z}\left[\frac{\Phi'_z}{\Phi'_w}\right] = \frac{\left[\Phi''_{z\bar z} + \Phi''_{z\bar w}\dfrac{\partial \bar w}{\partial \bar z}\right]\Phi'_w - \left[\Phi''_{w\bar z} + \Phi''_{w\bar w}\dfrac{\partial \bar w}{\partial \bar z}\right]\Phi'_z}{\Phi'^2_w} =$$

$$= \frac{\Phi'_w\left[\Phi''_{z\bar z}\Phi'_{\bar w} - \Phi''_{z\bar w}\,\Phi'_{\bar z}\right] - \Phi'_z\left[\Phi''_{w\bar z}\Phi'_w - \Phi''_{w\bar w}\,\Phi'_{\bar z}\right]}{\Phi'^2_w\,\Phi'_{\bar w}} =$$

$$= -\frac{\begin{vmatrix} 0 & \Phi'_{\bar w} & \Phi'_{\bar z} \\ \Phi'_w & \Phi''_{w\bar w} & \Phi''_{w\bar z} \\ \Phi'_z & \Phi''_{z\bar w} & \Phi''_{z\bar z}\end{vmatrix}}{\Phi'^2_w\,\Phi'_{\bar w}} = \frac{L(\Phi)}{\Phi'^2_w\,\Phi'_{\bar w}} = 0. \tag{2.49}$$

Thus we have shown that under our hypotheses conditions a) and b) are satisfied. In view of a) it follows that from (2.42) or, what is the same thing, from (2.43), there is defined a collection of functions $u = u(x, y, t)$ satisfying this system. Here t is an arbitrary constant of integration. This constant, by the theorem on the existence of solutions for a system of differential equations of type (2.42), may be assigned uniquely in such a way that the hypersurface $u = u(x, y, t)$ passes through some point (u_1, v_1, x_1, y_1) of the neighborhood U_P (here u_1, x_1, y_1 are given arbitrarily within that neighborhood, and v_1 in such a way that this point lies on the hypersurface $\Phi = 0$).

We choose hypersurfaces $u = u(x, y, t)$ passing through all the points of this neighborhood. They correspond to values of t for which $t_0 - \delta < t < t_0 + \delta$. Here t_0 is the value of the parameter corresponding to the hypersurface $u = u(x, y, t_0)$ passing through the point P, and δ is determined by the dimensions of the neighborhood U_P.

We also need to determine the function $v = v(x, y)$. To this end we make use of equation (2.40). It may be solved with respect to v, since $(\partial\Phi/\partial v)_P \neq 0$. Then, since the function $u(x, y)$ satisfies equations (2.43), for the value of $v(x, y, t)$ obtained from (2.40) we have equation (2.47), as was explained above. This means that the functions u and v satisfy the Cauchy-Riemann conditions. The derivatives of these functions are continuous by (2.43) and (2.47) and therefore in the neighborhood U_P the functions

$$\left.\begin{aligned} &W = u(x, y, t) + iv(x, y, t) = w(z, t), \\ &t_0 - \delta < t < t_0 + \delta \end{aligned}\right\} \tag{2.50}$$

are holomorphic functions of z and determine analytic surfaces. In the neighborhood U_P these surfaces consist of points of the hypersurface $\Phi = 0$, as follows from the method of definition of the function $v(x, y, t)$. Through each point of the hypersurface $\Phi = 0$, within the limits of the neighborhood U_P, there passes one such surface. With this our theorem is proved.

REMARK 1. In view of condition b) the function $u(x, y)$ is harmonic. Therefore the corresponding function $v(x, y)$ may be defined as the conjugate harmonic function. In making the definition of the function $v(x, y)$ in this way there is still one constant of integration that must be assigned in such a way that the analytic surface (2.50) passes through the point (u_1, v_1, x_1, y_1) of the surface $\Phi = 0$ belonging to U_P. If we find $v(x, y)$ as indicated above in the text of the proof of the theorem, we can avoid formulating condition b) separately and restrict ourselves to the statement that by (2.44), (2.45) and (2.49) it follows from the condition $L(\Phi) = 0$ that the system (2.42) is complete. However, the condition (2.44) turns out to be far more convenient than condition a).

REMARK 2. From our proof it follows that the representation $\Phi = 0$ in the form (2.50) is unique. This means that the analytic hypersurface cannot be divided into a collection of analytic surfaces in different ways, a result which also follows from the fact that two analytic surfaces intersect only in a finite set of points.

REMARK 3. In the case that $\Phi(u, v, x, y) \equiv \Phi(w, z)$ is indeed an analytic function of its variables, the segments of analytic surfaces defined (as indicated in the proof of the theorem) in the neighborhoods of a point P, are continuations of one another and form in their totality an entire analytic surface defined in the whole region of the hypersurface $\Phi = 0$ under discussion.

5. **Some theorems on analytically convex hypersurfaces.** Now we consider some special questions connected with the concept of analytic convexity. First of all we prove the following frequently applied theorem:

THEOREM 12.9. *The boundary of an ordinary bicylindrical region is analytically convex on both sides (i.e., if the boundary hypersurface $\Phi = 0$ is a twice continuously differentiable surface, then $L(\Phi) = 0$ and Φ is analytic).*

PROOF. Given a bicylindrical region $E = D \times G$, where D is a region in

the w-plane and G is a region in the z-plane. Suppose that $w_0 \in D$, $z_0 \in \partial G$. Then (w_0, z_0) is a point of the boundary of the region E. We consider some analytic surface passing through this point. The equation of this surface may be chosen in the form $z = f(w)$, where $f(w)$ is a series as indicated in Theorem 4.12 but containing fractional powers of $w - w_0$. By hypothesis $f(w_0) = z_0$. The function $z = f(w)$ will map the disk $|w - w_0| < \rho$ onto some neighborhood U_{z_0} of the point z_0 (this mapping of the disk $|w - w_0| < \rho$ onto U_{z_0} is not in general one-to-one). We choose ρ such that the disk $|w - w_0| < \rho$ is entirely inside the region D. Since z_0 is a boundary point of the region G, in the neighborhood U_{z_0} there will be points of G as well as points exterior to G. Suppose that z_1 is an interior point of G belonging to U_{z_0} and that z_2 is an exterior point also belonging to U_{z_0}. Since U_{z_0} is the image of the disk $|w - w_0| < \rho$ under its mapping by the function $z = f(w)$, there exist points w_1, w_2 in U_{z_0} such that $f(w_1) = z_1$, $f(w_2) = z_2$. But this means that on the analytic surface $z = f(w)$ in the neighborhood of the point (w_0, z_0) there are always both points (w_1, z_1) lying inside the region E and points (w_2, z_2) lying outside the region E. This proves our theorem.

Now we shall consider some differential-geometric properties of hypersurfaces and in particular the differential-geometric characterization of an analytic surface.

Suppose that the point $M(w, z)$ lies on the hypersurface $\Phi(w, z) = 0$, and that the original hypotheses concerning the function Φ are preserved. We consider three mutually perpendicular directions in the tangent plane at the point M, which we denote by T_1, T_2, N. Here T_1 and T_2 are two mutually perpendicular directions in the analytic plane lying in the tangent hyperplane and passing through M (such an analytic plane is defined uniquely; if the equation of the tangent hyperplane is represented in the form $\mathrm{R}(aW + bZ) = c$, then this plane will be given by the equation $a(W - w) + b(Z - z) = 0$), and N is the direction of the line in which the tangent hyperplane intersects an analytic plane passing through the normal to the hypersurface at the point M. We also construct points M', M'', M''' lying infinitely close to M, at a distance ds in the directions T_1, T_2, N on the hypersurface.[1] Furthermore, we consider the first and second

1) These three directions form the so-called "normal" system of coordinates in the hyperplane tangent to the hypersurface.

analytic angles $d\theta$ and $d\phi$ which are formed between the normals to the hypersurface at the points M and M', M and M'', M and M'''. Then we obtain six quantities:

$$\left[\frac{d\theta}{ds}\right]_{T_1}, \quad \left[\frac{d\phi}{ds}\right]_{T_1}, \quad \left[\frac{d\theta}{ds}\right]_{T_2}, \quad \left[\frac{d\phi}{ds}\right]_{T_2}, \quad \left[\frac{d\theta}{ds}\right]_{N}, \quad \left[\frac{d\phi}{ds}\right]_{N}, \tag{2.51}$$

characterizing the curvature of the hypersurface.[1)] They are closely connected with the components of the curvature tensor of the hypersurface. In addition, it turns out that the following relation holds:

$$\left[\frac{d\phi}{ds}\right]_N = \frac{1}{3}H + L(\Phi). \tag{2.52}$$

Here H is the mean curvature of the hypersurface at the point M. The analytic hypersurface is characterized by the equations

$$\left[\frac{d\theta}{ds}\right]_{T_1} = \left[\frac{d\theta}{ds}\right]_{T_2} = 0; \quad \left[\frac{d\phi}{ds}\right]_N = \frac{1}{3}H. \tag{2.53}$$

All of these equations are verified by direct calculation.

6. Sufficiency of the condition of analytic convexity. The theorems of Hartogs and Levi on the continuous distribution of singular and essentially singular points of holomorphic functions indicate only certain properties which sets consisting of such points must necessarily have. The following questions arise: is a region analytically convex at all of its boundary points necessarily a holomorphy region? Is it a meromorphy region?

The difficulty of the problem is caused by the fact that analytic convexity of the boundary of a region is a local property, in distinction from the property of being a holomorphically convex region considered in the preceding section. By changing only a small part of the boundary of a region, we may make the region at that place either analytically convex or nonconvex. Of course, those parts of the boundary of a region that are unaffected by this change remain, as before, analytically convex or nonconvex. On the other hand, the existence of functions holomorphic or meromorphic in some region and continuable to its boundary is connected with the structure of the entire region in the large. Thus the problem consists of establishing relations between quite different properties of the region.

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1) These quantities and equation (2.52) were obtained by I. Mitrohin [1].

The first step toward the solution of this problem was taken by E. E. Levi [2]. He proved in that paper the following two theorems.

THEOREM 12.10. *Suppose that at every point of some hypersurface* $\Phi = 0$ *of the class* $\mathcal{C}^2$ *the following relation holds:* $L(\Phi) = 0$. *Then for every ordinary point of this hypersurface one can find a neighborhood of it such that the hypersurface* $\Phi = 0$ *is, within the limits of this neighborhood, the natural boundary for some meromorphic (in particular, holomorphic) function on the side* $\Phi > 0$ *(and on the side* $\Phi < 0$*).*

THEOREM 12.11. *Suppose that at an ordinary point of some hypersurface* $\Phi(w, z) = 0$ *of class* $\mathcal{C}^2$ *the relation* $L(\Phi) > 0$ *holds. Then one can find a neighborhood of this point such that the hypersurface* $\Phi = 0$ *is, within the limits of this neighborhood, the natural boundary for some function which is meromorphic (in particular, holomorphic) on the side* $\Phi < 0$ *lying within this neighborhood.*

These theorems establish the fact that small pieces of hypersurfaces of class $\mathcal{C}^2$ which are analytically convex in the sense of Levi are natural boundaries of meromorphic (in particular, holomorphic) functions. It follows from these theorems that if $D \subset C^2$ is a region bounded by a hypersurface Γ of class $\mathcal{C}^2$ which is everywhere analytically convex in the sense of Levi, then for each boundary point P of this region one can construct a function $f_P(w, z)$ which is holomorphic in the part of some neighborhood of the point P lying in D and which has the hypersurface Γ as its natural boundary within the limits of this neighborhood. However, this theorem leaves completely open the question as to whether there exists a function holomorphic or meromorphic in the region D and having all the points of its boundary Γ as singular points.

An affirmative answer to this question, bringing to an end more than thirty years of investigation, appeared in 1942 in the paper [3] of K. Oka. In his paper Oka considered a region D of the space of two complex variables. The general case of a space of $n \geq 2$ variables was considered in later papers by Oka himself and also by Bremermann and Norguet. [1)] The final result runs as follows:

THEOREM 12.12. *A region* $D \subset C^n$ *which is analytically convex in the sense of Hartogs is a holomorphy region at all points of its boundary.*

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1) See Oka [5, 6], Bremermann [1], Norguet [1]. See also Fuks [2].

Hence for regions D of C^2, namely the space of two complex variables, it is easy to show that the following is true:

THEOREM 12.12_1. *A region $D \subset C^2$ bounded by the hypersurface $\Phi = 0$ of class $\mathcal{C}^2$ (the values $\Phi < 0$ correspond to the interior of the region D) on which $L(\Phi) \geq 0$ is a holomorphy region.*

From Theorems 11.4 and 11.6 on the one hand, and Theorem 12.12 on the other, it results that a single-sheeted region is a holomorphically convex region if and only if it is analytically convex in the sense of Hartogs. In the process of proving Theorem 11.4 we constructed for any holomorphically convex region D a function having that region as its holomorphy region. Therefore, from the fact that a single-sheeted meromorphy region is analytically convex it follows that such a region is always a holomorphy region.

Detailed consideration of these questions, and also the extension of the concept of analytic convexity in the sense of Levi to the case of a space of $n > 2$ complex variables, is contained in the second volume of the present book.

7. Convergence and normality regions. The role of regions analytically convex in the sense of Hartogs and Levi was substantially enlarged when in 1926 G. Julia [1] published his investigations connected with regions of uniform convergence of sequences and normality regions of sets of holomorphic functions.

In his study of these regions, G. Julia established that an arbitrarily chosen region D over the space of $n \geq 2$ complex variables is not necessarily either a region of uniform convergence (of the first or second kind) or a normality region (of the first or second kind).

DEFINITION (region of uniform convergence). A region D over the space C^n is said to be a *region of uniform convergence of the first kind* (*second kind*) if

1) there exists a sequence of functions $\{f_\nu\}$, $\nu = 1, 2, \cdots$, that are holomorphic and uniformly convergent (uniformly converging to ∞) in the region D;

2) there does not exist a larger region containing the region D and possessing the property 1) for the sequence $\{f_\nu\}$.

DEFINITION (normality region). The region D over the space C^n is said to be a *normality region of the first kind* (*second kind*) if

1) there exists a set of functions $\{f\}$ holomorphic in the region D and forming there a normal family of the first kind (second kind);

2) there does not exist a larger region containing the region D and

possessing property 1) with respect to the set of functions $\{f\}$.

The set $\{f\}$ is said to form a *normal family* in the region D if from each infinite subset of $\{f\}$ one may select a sequence which uniformly converges in the region D. If among these sequences there are some that converge to an identical infinity, then $\{f\}$ is said to be a normal family of the second kind, and if there are no such sequences, then it is of the first kind.

G. Julia further proved that single-sheeted regions of uniform convergence of the first and second kinds and normality regions of the first and second kinds are analytically convex in the sense of Hartogs. He also proved that a hypersurface of class $\mathcal{C}^2$ entering into the boundary of such a region is convex (from outside) in the sense of Levi. G. Julia succeeded also in extending to these regions certain other properties of holomorphy regions and meromorphy regions obtained in their time by P. Hartogs and E. E. Levi.

These results of Julia led to the so-called "Julia hypothesis" that uniform convergence regions of the first kind and normality regions of the first kind are holomorphy regions, and that those of the second kind are meromorphy regions. The confirmation of this hypothesis was obtained by H. Cartan and P. Thullen [1] in 1932.

The results of Julia concerning regions of uniform convergence are close to the properties of convergence regions of certain special series (power series and those arising from power series as a result of certain transformations) found already quite a long time ago by P. Hartogs [1] and other mathematicians.

A detailed discussion of these questions is contained in the second volume of the present book (to be published).

8. Analytically convex covering regions. We extend the concept of analytic convexity to covering regions over the space C^n.[1] To this end we first give the following definition.

DEFINITION (distinguished family of analytic sets). In the extended region $\breve{D} = D + \partial \breve{D}$ over the space C^n we consider a family of one-complex-dimensional analytic sets $G = \{G(w, t),\ 0 \leq t \leq 1\}$, each of which is defined by a holomorphic mapping of the closed disk $|w| \leq 1$ into $\breve{D}$. We assume that this mapping depends continuously on the parameter t in the given closed interval. We shall denote the sets described above by the symbols G_η and $G_\eta(w, t)$, if

1) See Grauert-Remmert [1].

the set of geometric points $\bigcup_{0 \le t \le 1} G(w, t)$ is contained in some hyperball of radius η.

A family G of this sort is said to be *distinguished* if the sets $G(w, t)$ for $0 \le t \le 1$, $|w| < 1$ and for $0 \le t \le 1$, $|w| = 1$ are contained in the region D.

Now we may formulate the following definition.

DEFINITION (analytically convex region over the space C^n). The region D over the space C^n is said to be analytically convex at its boundary point $\breve{r} \in \partial D$ if there exists an extended neighborhood $\breve{V}_D(r)$ and a number $\eta > 0$ such that each distinguished family of analytic sets G_η contains no points belonging to the set $\breve{V}_D(r) \cap \partial \breve{D}$ (i.e., a family G_η containing points of this set cannot be distinguished).

It is easy to see that in the case of a single-sheeted region this definition is essentially equivalent to the necessary and sufficient condition for analytic convexity contained in Theorem 12.1.

This definition is frequently replaced by others, which are essentially equivalent to it.[1] We note that the various variations of the theorem on the continuous distribution of singular points (see, for example, Theorems 11.9 and 11.10) always contain some characterization of the natural boundary of the functions. Each of them may be used as the basis of the definition of analytic convexity.

We note among these a number of definitions due to Henri Cartan [1].

The region D is said to be analytically convex at its boundary point $\breve{r} \in \partial D$ if there exists a neighborhood $\breve{V}_D(r)$ adherent to the point r which is a holomorphy region.

The region D is said to be analytically convex if it is analytically convex at all of its boundary points.

To K. Oka is due the following fundamental theorem.

THEOREM 12.13. *The non-interiorly-branching region D over the space C^n is a holomorphy region if and only if it is analytically convex.*

The question as to the possibility of extending this theorem to infinite regions (regions over the space P^n) and to interiorly-branching regions remains open at the present time. However, it is known that Theorem 12.13 does not

1) See Docquier-Grauert [1].

extend to regions which are simultaneously infinite and interiorly-branching. [1)]

§13. HOLOMORPHY HULLS. REGIONS WITH AUTOMORPHISMS

1. Fundamental properties.

DEFINITION (holomorphy hull of a region). A region which is the intersection of the holomorphy regions of all the functions holomorphic in the region D (relative to the region D) is called the holomorphy hull of the region D and is denoted by $H(D)$ (in what follows we shall frequently, for abbreviation, call the region $H(D)$ the hull of the region D).

From this definition it is evident that every region D of the space P^1 coincides with its holomorphy hull. Therefore, the concept of holomorphy hull has no meaning in the theorem of functions of one variable. It plays an important role in the general theory of functions of several complex variables.

From the definition of holomorphy hull given above it follows that every function which is holomorphic in some region D may be analytically continued to the region $H(D)$. Every other region with this property will contain the region $H(D)$ in its interior.

Evidently, if the region $D_1 < D_2$, then [2)]

$$H(D_1) < H(D_2). \tag{2.54}$$

From Theorem 11.7 immediately follows the so-called *fundamental theorem on holomorphy hulls.*

THEOREM 13.1. *The holomorphy hull $H(D)$ of any region D is a holomorphy region for some function.*

We turn our attention now to the properties of holomorphy hulls.

THEOREM 13.2. *Every meromorphic function $f(z)$ in the region D takes on in the region $H(D)$ only those values which it takes on in the region D.*

PROOF. If $f(z) \neq a$ in the region D, then the function $[f(z) - a]^{-1}$ is

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1) See Docquier-Grauert [1].

2) The region $H(D_1)$ is obtained by taking an intersection: 1) one intersects the holomorphy regions of functions holomorphic in the region D_2, yielding as the result $H(D_2)$, and in addition 2) the holomorphy regions of functions holomorphic in the region D_1 but not in the region D_2.

holomorphic in the region D, which means that it is also holomorphic in the region $H(D)$. Therefore it follows that the function $f(z)$ is also meromorphic in the region $H(D)$ and does not take on there the value a. Thus if the meromorphic function $f(z)$ omits some value a in the region D, then in addition to the assertion of our theorem, we may also keep in mind that the function $f(z)$ is meromorphic in the region $H(D)$.

It also follows that we always have

$$\sup |f(D)| = \sup |f(H(D))|. \tag{2.55}$$

This last equation makes it easy to prove the following proposition:

THEOREM 13.3. *The holomorphy hull of a bounded region is again a bounded region.*

We consider in the regions D and $H(D)$ the functions z_k, $k = 1, \cdots, n$, and make use of equation (2.55) and the boundedness of the region D. Then we obtain

$$\sup_{z \in H(D)} \sum_{k=1}^{n} |z_k^2| \leq \sum_{k=1}^{n} \sup_{z \in H(D)} |z_k^2| = \sum_{k=1}^{n} \sup_{z \in D} |z_k^2| < \infty. \tag{2.56}$$

Therefore our assertion follows.

THEOREM 13.4. *Suppose that the region $D_0 \ll D$ and r is the minimal limiting distance of the region D_0 in the region D. Then the minimal limiting distance of the region $H(D_0)$ in the region $H(D)$ is not less than r.*

PROOF. As we have shown above, if $f(z)$ is holomorphic in the region D_0, then

$$\sup |f(D_0)| = \sup |f(H(D_0))|.$$

Therefore, if M is any finite point of the region $H(D_0)$, and $f(z)$ is a function holomorphic in the region D (and accordingly also in the region $H(D)$), then

$$|f(M)| \leq \sup |f(D_0)|. \tag{2.57}$$

The same inequality holds if in (2.57) we replace the point M by a point M^* (the point M^* corresponds to the point M in the region $H(D)$ because of the relation $H(D_0) > H(D)$), and the region D_0 by the region D_0^* (D_0^* is a subregion of $H(D)$ which covers the region D_0 in view of the fact that $H(D_0) < H(D)$). Then by the first part of Theorem 11.1 the function $f(z)$ will be holomorphic in the polycylinder $S(M^*, \rho)$, where ρ is the minimal limiting distance of the region

D_0^* in the region $H(D)$, or, what is the same thing, of the region D_0 in the region $H(D)$. Evidently $\rho \geq r$ (since $D < H(D)$). Therefore it follows that any finite point M of the region $H(D_0)$ must have a limiting distance in the region $H(D)$ not less than r, since otherwise all the functions $f(z)$ in question would be holomorphic in the region D (but among them there is at least one for which the region $H(D)$ is a holomorphy region), and consequently also in the polycylinder $S(M^*, r)$, and these functions would therefore be continuable to the limits of the region $H(D)$.

Thus it is proved that all the finite points M of the region $H(D_0)$ have a minimal limiting distance in the region $H(D)$ not less than r.

So the proof of the theorem is complete.

Recalling the remark made earlier concerning an interior region with a minimal distance from the boundary of an exterior region (see subsection 3 of §8), we arrive at the following corollary of the theorem just proved:

If the region $D_0 \ll D$ *and the region* $H(D_0)$ *is finite-sheeted, then the region* $H(D_0) \ll H(D)$.

The following theorem follows immediately from the theorem proved above.

THEOREM 13.5. *Let* D_r *be a set of points of a finite region* D *whose limiting distance in the region* D *is larger than a definite number* r.

Then the region D_r *is either empty, or else decomposes into a series* $D_r^{(1)}, D_r^{(2)}, \cdots$ *of subregions of the region* D. *If* D *is a holomorphy region for some function, then also all the regions* $D_r^{(k)}$ *are holomorphy regions.*

(Otherwise the region $H(D_r^{(k)}) \neq D_r^{(k)}$ would have a limiting distance in the region $H(D) = D$ less than r, which is excluded by the preceding theorem.)

THEOREM 13.6. *Suppose that* D *is any region and* $H(D)$ *is its holomorphy hull. If* $f(z)$ *is a function holomorphic in the region* $H(D)$ *and having some boundary point* Q *of the region* $H(D)$ *as a singular point, then the function* $f(z)$ *also has a singular point at some boundary point of the region* D.

PROOF. Denote by E the holomorphy region of the function $f(z)$. If our theorem were not true, then the region $D \ll E$ and there would exist a minimal limiting distance r of the region D in the region E with $r > 0$. Since the region $H(E) = E$, it follows from Theorem 13.4 that the minimal limiting distance of the region $H(D)$ in the region E is not less than r. This contradicts our hypothesis about the point Q. Thus the theorem is proved.

From the theorem just proved it follows that *if to each boundary point Q of the holomorphy region D there corresponds a function $f(z)$ which is holomorphic at all points of the closed region $\overline{D}$ except at the point Q, then there does not exist a subregion $D_0 \neq D$, $\partial D_0 \cap \partial D \neq \phi$, for which $H(D_0) = D$.* For example, if D is the hyperball $|w|^2 + |z|^2 < 1$, then, as we have already seen above, for each point $Q(w_0, z_0)$ of its boundary one may find a function $1/(\overline{w}_0 w + \overline{z}_0 z - 1)$ holomorphic at all the points of the closed hyperball $\overline{D}$ except the point (w_0, z_0) itself. Thus, a hyperball is not a regularity hull for any of its subregions.

Another example of a holomorphy region with the same properties is provided by the bicircular region $\{A|w|^a + B|z|^2 < 1\}$, where $a > 0$, and A, B are real. Examples of regions which are holomorphy hulls for their subregions were given in subsection 3 of this section.

REMARK. Along with holomorphy hulls for finite regions it is useful in a number of cases to consider K-convex hulls. By a *K-convex hull* of the region D we understand the intersection of all K-convex regions containing the region D in their interiors. This will be denoted by the symbol $K(D)$.

Since the intersection of K-convex regions is again K-convex, the region $K(D)$ is the smallest K-convex region containing the region D.

The region $K(D)$ always exists, since the set of K-convex regions containing the region D is never empty (for example, the intersection Δ of the holomorphy regions of the functions constituting the class K belongs to the region $K(D)$).

2. Mappings of holomorphy regions and holomorphy hulls.

THEOREM 13.7. *A region E which is the image of a holomorphy region D under a bimeromorphic (in the narrow sense) mapping T is itself a holomorphy region.*

This theorem is almost obvious.

If $f(z)$ is a function for which the region D is a holomorphy region and $z = z(Z)$ is the mapping inverse to the mapping T, then the function $f[z(Z)] = \phi(Z)$ will have the region E as a holomorphy region. From Theorems 11.4, 11.6 and 13.7 there follows the following:

COROLLARY. *The property of a region of being holomorphically convex is invariant under bimeromorphic mappings.*

As we have already noted, Theorem 11.4, generally speaking, does not hold

for interiorly-branching regions. Therefore this last corollary, generally speaking, does not hold for generalized biholomorphic mappings of the corresponding type (considered in subsection 4 of §10).

THEOREM 13.8. *If* $T\{Z = Z(z)\}$ *is a biholomorphic mapping of the region* D *onto some region* E, *then this same mapping, continued to the region* $H(D)$, *is a biholomorphic mapping of that region* $H(D)$ *onto the holomorphy hull* $H(E)$.

PROOF. Evidently the functions $Z_k(z)$ $(k = 1, \cdots, n)$, holomorphic in the region D, may be analytically continued to the region $H(D)$. If their Jacobians are different from zero in the region D, they will remain different from zero also in the region $H(D)$. By Theorem 10.1 they biholomorphically map the region $H(D)$ onto some holomorphy region $G > E$. If $G = H(E)$, then our theorem is proved. In the contrary case $E < H(E) < G$. Then the inverse mapping T^{-1} maps the region G onto the region $H(D)$ and the region $H(E)$ onto some region B, with $D < B < H(D)$ and $B \neq H(D)$.

This last inequality is impossible, since the region $H(D)$ is the intersection of the holomorphy regions for all the functions holomorphic in the region D. Therefore this region is the smallest holomorphy region containing the region D.

DEFINITION (automorphism of a region). An automorphism of a region is a generalized biholomorphic mapping of the region onto itself.

It follows from Theorem 13.8 that among the automorphisms of a holomorphy region D there belong those biholomorphic automorphisms of all the regions E for which the region D is a holomorphy hull, i.e., for which $H(E) = D$. The automorphisms of such a region D constitute the set of all automorphisms of the regions E. Therefore it follows that holomorphy regions having no automorphisms (such regions are said to be *rigid regions*) must appear relatively seldom. The first example of a rigid (and therefore simply-connected) region was constructed in 1935.[1)]

3. Disk-shaped regions and their holomorphy hulls. In subsection 3 of §3 of Chapter I we considered n-circular regions of the space C^n. Among their number appear the convergence regions for power series. Now we give a definition.

DEFINITION (n-circular region over the space C^n). The region D over the space C^n is said to be an n-circular region with center at the point M_0 with coordinates $a_1, \cdots, a_n$ if its automorphisms consist of the n-parameter group

1) See Behnke-Peschl [1].

$\{T(\theta_1, \cdots, \theta_n)\}$. Here $T(\theta_1, \cdots, \theta_n) = T(\theta)$ is a generalized biholomorphic mapping carrying the point $P_0(z_1^{(0)}, \cdots, z_n^{(0)})$ into the point $P(z_1, \cdots, z_n)$:

$$T(\theta_1, \cdots, \theta_n) \quad z_j = (z_j^{(0)} - a_j)\, e^{i\theta_j} + a_j, \tag{2.58}$$

where $0 \leq \theta_j \leq 2\pi$, $j = 1, \cdots, n$.

We note that if the region D is multiple-sheeted, then in order to define the mapping $T(\theta)$ it is necessary to indicate, besides the relations (2.58), the order of the correspondence between *distinct* analytic points with the same coordinates and their images. This relation must be one-to-one (since every automorphism has this property). An n-disked region may be an interiorly-branching region. For this reason, in our definitions we speak of a *generalized* biholomorphic mapping.

The intersection of an n-circular region with planes parallel to the coordinate planes z_j $(j = 1, \cdots, n)$ is either empty or consists of (generally speaking, multiple-sheeted) circular annuli. In a special case this annulus may reduce to a disk. If this last happens for every such plane, then the corresponding n-circular region is said to be *complete*. A complete n-circular region is always star-shaped and single-sheeted; along with the point $M_1(z_1^{(1)}, \cdots, z_n^{(1)})$ the closed polycylinder

$$\{\, |\, z_j - a_j \,| \leq |\, z_j^{(1)} - a_j \,|, \ \ j = 1, \cdots, n\}$$

must necessarily belong to it.

From Abel's Theorem 3.4 it follows that a function $f(z)$ which is holomorphic in some n-circular region D to which its center belongs must necessarily be analytically continuable to the smallest complete n-circular region containing the region in question. Now we may formulate a stronger result. Such a function $f(z)$ may be continued to the smallest n-circular region analytically convex in the sense of Hartogs and containing the given region D. The region obtained in this way is a holomorphy hull $H(D)$ for the region D. The uniformly convergent power series representing the function $f(z)$ in the region D may be continued to the region $H(D)$.

If the bicircular region $D = \{\, |\, z \,| - \phi(\, |\, w \,|\,) < 0\}$ in the space of complex variables w, z is bounded by the hypersurface $\Phi = |\, z\,| - \phi(\,|\, w\,|\,) = 0$ belonging to the class $\mathcal{C}^2$, then at the points of this hypersurface

$$L(\Phi) = -\frac{1}{16}\left[\frac{d^2\phi}{dr^2} - \frac{1}{\phi}\left(\frac{d\phi}{dr}\right)^2 + \frac{1}{r}\left(\frac{d\phi}{dr}\right)\right] = -\frac{1}{16}\,\frac{d^2(\ln r')}{(d\ln r)^2},$$

where $r = |w|$, $r' = |z|$. Thus, if at the points of the hypersurface $\Phi = 0$ we have $d^2(\ln r')/(d\ln r)^2 < 0$, then this hypersurface will be analytically convex on the side $\Phi < 0$, and D will be a holomorphy region. This is the so-called condition of logarithmic convexity. Hartogs in his day discovered that if this condition is satisfied the bicircular region D is a region of uniform convergence for the power series.

So the following theorem is valid.

THEOREM 13.9. *The holomorphy hull of an n-circular region to which its center belongs and is not an interior branch point is the smallest analytically closed n-circular region containing the given region.*

On the basis of this theorem, we may construct a multiple-sheeted region which has a single-sheeted holomorphy hull.

Consider[1] a region D over the space C^2 of variables w, z, which is the union of sets

$$\left.\begin{array}{l} D_1\left\{\left[|w| < 1,\ |z| < \frac{1}{4}\right] \cup \left[\frac{3}{4} < |w| < 1,\ \frac{1}{4} \le |z| < 1\right]\right\}, \\[2ex] D_2\left\{\left[\frac{1}{4} < |w| \le \frac{3}{4},\ \frac{3}{4} < |z| < 1\right] \cup \left[\frac{1}{4} < |w| < \frac{1}{2},\ |z| \le \frac{3}{4}\right]\right\}. \end{array}\right\} \tag{2.59}$$

Here the points of the sets D_1 and D_2 lying over the region $\{1/4 < |w| < 1/2,$ $|z| < 1/4\}$ are considered as distinct points of the region D with the same projections.

Thus D is a multiple-sheeted region; its holomorphy hull is the bicylinder $\{|w| < 1,\ |z| < 1\}$. In fact, every function holomorphic in the region D is also holomorphic in a subregion D_1 of it, and accordingly the function may be represented by a double power series converging in the entire bicylinder $\{|w| < 1,\ |z| < 1\}$.

Now we consider disk-shaped regions of other kinds.

DEFINITION ($(p_1, \cdots, p_n)$-circular region over the space C^n). A region D over the space C^n is said to be a $(p_1, \cdots, p_n)$-circular region with center at

1) See Bremermann [4].

the point M_0 with coordinates $(a_1, \cdots, a_n)$ if the set of its automorphisms is the one-parameter group $\{T(p_1\theta, \cdots, p_n\theta)\}$. Here $T(p_1\theta, \cdots, p_n\theta)$ is a generalized biholomorphic mapping carrying the point $P_0(z_1^{(0)}, \cdots, z_n^{(0)})$ into the point $P(z_1, \cdots, z_n)$:

$$T(p_1\theta, \cdots, p_n\theta) \qquad z_j = (z_j^{(0)} - a_j)\, e^{ip_j\theta} + a_j, \tag{2.60}$$

where $0 \leq \theta \leq 2\pi$, $j = 1, \cdots, n$ and $p_1, \cdots, p_n$ are relatively prime integers. We note that if the region D is multiple-sheeted, then in order to define the transformation $T(p_1\theta, \cdots, p_n\theta)$ it is necessary, besides relations (2.60), to indicate the order of correspondence between distinct various analytic points with the same coordinates and their images.

In what follows we restrict ourselves to the case of two complex variables w and z.

For $m = p = 1$ an (m, p)-circular region is called simply a *disk* region. If the intersection of a disk region with each analytic plane $z - z_0 = t(w - w_0)$ is a complete disk, then the disk region is said to be complete. A complete disk region is always star-shaped and single-sheeted; along with the point $M_1(w_1, z_1)$ there necessarily belong to it all the points with coordinates $(w_0 + t(w_1 - w_0),\ z_0 + t(z_1 - z_0))$, for $|t| \leq 1$ (making up a complete disk).

For $m = 1$, $p = 0$ or $m = 0$, $p = 1$ an (m, p)-circular region is said to be a *semidisk region* with symmetry plane $w = w_0$ (or correspondingly $z = z_0$). If the intersection of the semidisk region for $m = 1$, $p = 0$ with each plane $z =$ const (respectively for $m = 0$, $p = 1$ with each plane $w =$ const) is either empty or is a complete disk, then the semidisk region is said to be complete. A complete semidisk region may fail to be single-sheeted.

We state the following theorem without proof.

THEOREM 13.10. *The holomorphy hull of a bounded disk (semidisk) region, to which its center belongs and is not an interior branch point (and which has interior points in the symmetry plane), is the smallest analytically convex disk (semidisk) region containing the given region.*

Theorem 13.10 for single-sheeted regions is a corollary of Theorems 12.13 and 13.8.

An analogous proposition may be stated also in the general case of (m, p)-circular regions.

Disk-shaped regions were studied in detail in the papers of Hartogs, Behnke, Faber, H. Cartan, Thullen and several other mathematicians. [1]

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1) See Hartogs [1], Cartan [2, 3], Faber [1]. The theory of disk-shaped regions was more completely presented in the first edition of the present book.

4. **Tubular regions and their holomorphy hulls.**

DEFINITION. The region S of the space C_z^n of variables $z_j = x_j + iy_j$, $j = 1, \cdots, n$ is said to be *tubular* (or tube-shaped, or cylindrical) if along with each point $(z_1^{(0)}, \cdots, z_n^{(0)})$ of this region there also belong to it all the points $(z_1^{(0)} + \rho_1, \cdots, z_n^{(0)} + \rho_n)$, where ρ_j $(j = 1, \cdots, n)$ are arbitrary real numbers.

Thus such a region S has an n-parameter group of automorphisms

$$(T) \qquad z_j = z_j^{(0)} + \rho_j \quad (j = 1, \cdots, n). \tag{2.61}$$

The mapping

$$(\Phi) \qquad w_j = e^{iz_j}, \quad j = 1, \cdots, n, \tag{2.62}$$

carries the tubular region S into an n-circular region $D = \Phi S$ of the space C_w^n of variables $w_1, \cdots, w_n$ with center at the origin of coordinates, namely the point O.

The points of a tubular region S lying in the subspace $R_n^{(\mathrm{Im})}\{x_1 = 0 \cdots = x_n = 0\} \subset C^n$ form a region $S^{(\mathrm{Im})}$ called its *base*. The region $S^{(\mathrm{Im})}$ plays the same role in the theory of tubular regions as the region D^+ (the image of an n-circular region D in the absolute octant R_n^+) in the theory of n-circular regions. If $D = \Phi S$, then $D^+ = \Phi S^{(\mathrm{Im})}$.

We note that under the mapping Φ every tubular region S with a bounded base goes into an n-circular region not containing the point O. If we continue the mapping (2.62) in an appropriate way to the points at infinity of the space $G^n \supset C_z^n$ and then apply it to the tubular region S with an unbounded base, we may obtain as its image an n-circular region containing its center inside or on its boundary.

A tubular region S is said to be octant-shaped or complete if along with each point $(z_1^0, \cdots, z_n^0)$ it contains all the points $(z_1, \cdots, z_n)$ for which $\operatorname{Im} z_j \geq \operatorname{Im} z_j^0$, where $j = 1, \cdots, n$. To an octant-shaped tubular region S corresponds a complete n-circular region D with center at the origin of coordinates.

Under the mapping Φ the boundary $\partial S^{(\mathrm{Im})}$ of the region $S^{(\mathrm{Im})}$ goes into the boundary ∂D^+ of the region D^+. As noted above, the bicircular region $D \subset C_w^2$ bounded by the hypersurface $|w_2| = \phi(|w_1|)$ is analytically convex if the boundary ∂D^+ of the region D^+ in the absolute 4-plane is logarithmically convex, i.e., if $d^2 \ln |w_2| / (d \ln |w_1|)^2 < 0$. As a result of the substitution (2.62) this inequality takes the form $d^2 y_2 / dy_1^2 > 0$. However, one cannot yet conclude that a tubular region S is analytically convex if its base is bounded by a curve on which $d^2 y_2 / dy_1^2 > 0$ (if this condition is satisfied the curve is concave, and the region bounded by it is convex), since the mapping Φ is not a biholomorphism.

Nevertheless this conclusion turns out to be valid, even in the general case of the space of $n > 1$ complex variables. *A tubular holomorphy region in the space* C_z^n *is a tubular region with a convex base* $S^{(\mathrm{Im})}$. The construction of holomorphy hulls for the tubular region S is equivalent to the construction of the smallest convex hull $\widetilde{S}^{(\mathrm{Im})}$ of its base $S^{(\mathrm{Im})}$ and thence of the tubular region $\widetilde{S}$ with the base $\widetilde{S}^{(\mathrm{Im})}$. 1)

Because of the fact that the inverse mapping Φ^{-1} is not univalent to a holomorphic function $f(z)$ defined in a tubular region S there corresponds in an n-circular region a holomorphic function $f \circ \Phi^{-1}$ only in the case that

$$f(z_1 + 2\pi N_1, \cdots, z_n + 2\pi N_n) = f(z),$$

i.e., if this function f has period 2π in every variable z_j.

On the other hand, to each function $\phi(w)$ that is holomorphic in the n-circular region D there corresponds a function $f(z) = \phi \circ \Phi$ that is holomorphic in the tubular region $S = \Phi^{-1} D$. One may show the following: if D is a holomorphy region, then $S = \Phi^{-1} D$ is a holomorphy region. The converse generally speaking does not hold.

5. **Example of a single-sheeted region with a multiple-sheeted holomorphy hull.** An example of this sort was first constructed by Thullen [1]. Consider the following semidisk region D of the space C^2 of variables w, z. Its projection onto the coordinate plane w is the ring $\frac{1}{2} \leq |w| < 1$. On each plane $w = re^{i\theta}$, where $\frac{1}{2} < r < 1$, those points belong to D whose coordinates z satisfy the condition

$$\max\left[-\frac{1}{2} + \theta,\ 0\right] \leq |z| < \theta + \frac{1}{2}, \tag{2.63}$$

where we take the size of the angle θ to satisfy $0 \leq \theta \leq 2n\pi$ (here n is a positive integer).

The region thus obtained evidently admits automorphisms $W = w$, $Z = ze^{i\theta}$ and accordingly is semidisked (with symmetry plane $z = 0$). Considering the coordinates z of the points of the region D corresponding to the values of the region w on the disk $w = re^{i\theta}$ under continuous variation of θ from 0 to $2n\pi$ ($n > 1$), we find that D is a spiral-shaped region. The region D contains points lying in the plane $z = 0$ and therefore its holomorphy hull $H(D)$ will be a complete semidisk region containing the given region. We put into correspondence with the point $(w = re^{i\theta}, z)$ of the region $D(0 \leq \theta \leq 2n\pi$, z defined from condition (2.63)) the value of the function $f(w, z) = \ln|w| + i\theta$. The function $f(w, z)$

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1) See Bochner-Martin [1] or the second volume of the present book (to be published).

defined in this way in the region D will be univalent and holomorphic there. It must therefore be univalent and holomorphic in the region $H(D)$. In each plane $w = re^{i\theta}$, there must belong to the region $H(D)$ at least the entire disk $|z| \leq \theta + \frac{1}{2}$ and therefore the point $z = 0$. At the geometrical point $(re^{i\theta}, 0)$ lying in the region $H(D)$, the function $\ln|w| + i\theta$ will for $\theta = 2\pi\nu i$ $(\nu = 1, \cdots, n)$ take on n distinct values. Therefore it follows that in the region $H(D)$ over this point there must lie not fewer than n analytic points. The region $H(D)$ must be at least n-sheeted.

6. Holomorphy hull of the product of regions. First of all we formulate the following lemma.

LEMMA. *Let D be a region over the space C_w^m of variables $w_1, \cdots, w_m$, and let E_1 and E_2 be regions over the space C_z^n of variables $z_1, \cdots, z_n$, with $E_1 \subset E_2$. If the function $f(w, z)$ is holomorphic in the region $D \times E_1$ and holomorphic in z in the region E_2 for each fixed value of $w \in D$, then this function is holomorphic in the set of its variables in the region $D \times E_2$.*

This lemma is an extension to the case of regions over the space C^n of complex variables of the principal theorem of Hartogs, presented in subsection 3 of §1 of Chapter I. We will not give its proof here.

THEOREM 13.11.[1] *If D and $H(D)$ are regions over the space C_w^m of variables $w_1, \cdots, w_m$, and E and $H(E)$ are regions over the space C_z^n of variables $z_1, \cdots, z_n$, then*

$$H(D \times E) = H(D) \times H(E). \tag{2.64}$$

PROOF. Each function $f(w, z)$ holomorphic in the region $D \times E$ is holomorphic in w in the region D for each fixed z in the region E. This function will be holomorphic in w also in the region $H(D)$. Therefore, from the preceding lemma, it follows that the function $f(w, z)$ is holomorphic in the set of its variables in the region $H(D) \times E$. By repeating the argument, we establish that the function $f(w, z)$ is holomorphic also in the region $H(D) \times H(E)$. Since $f(w, z)$ is an arbitrary function holomorphic in the region $D \times E$, it follows that

$$H(D \times E) > H(D) \times H(E). \tag{2.65}$$

On the other hand, the holomorphy hull $H(D \times E)$ is the smallest holomorphy region which contains the region $D \times E$. Therefore

$$H(D \times E) < H(D) \times H(E). \tag{2.66}$$

Our assertion follows from the relations (2.65) and (2.66).

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1) See Bremermann [2], Sommer-Mehring [1].

CHAPTER III

COMPLEX SPACES

§14. COMPLEX ANALYTIC MANIFOLDS. COMPLEX ANALYTIC COVERINGS

1. Complex manifolds. Complex analytic manifolds or, briefly, complex manifolds, are particular cases of the manifolds considered in the Introduction. For a complex manifold X the charts (U_j, ψ_j) that make up its complete atlas must be holomorphically compatible with one another. This means that if the intersection of the elements U_i and U_j of the manifold is not empty, the homeomorphism $\psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$ is biholomorphic. Every (connected) component of a complex manifold has an even topological dimension equal to twice its complex dimension. If all components of a manifold Z have the same dimension n, the manifold is said to be homogeneous or pure-dimensional, and is denoted by Z_n.

We note that in considering complex manifolds we usually take as the region $\psi_j U_j$ the polycylinder $\{|t_i| < 1;\ i = 1, \cdots, n_j\}$ in the space $C_t^{n_j}$ of the complex variables $t_1, \cdots, t_{n_j}$. The latter are called (locally) uniformizing parameters or coordinates of points in the element U_j.

Examples of complex manifolds are: all open sets of the spaces C^n and P^n $(n \geq 1)$, analytic sets in these spaces which consist of ordinary points, and plane regions of a covering without inner branch points. Clearly, if we add to a plane region of a covering its complex uniformizing branch points we again have a complex manifold. On the other hand, by adding to a suitable region its complex non-uniformizing branch points, we obtain a set that is not a complex manifold.

DEFINITION (holomorphic function on a complex manifold). A function f, defined in the neighborhood of the point z of a complex manifold Z, is said to be holomorphic at that point if there exists a neighborhood $V_z \subset U_j$ of z, where U_j is an element of the manifold Z, such that the function $f \circ \psi_j^{-1}$ is holomorphic in the region $\psi_j^{-1} V_z$ of the auxiliary space of uniformizing parameters.

The exterior form $\alpha = \Sigma_p^n A_{i_1 \cdots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}$, defined in some neighborhood of the point $z \in Z$, is said to be holomorphic at z if the functions $A_{i_1 \cdots i_p}(z)$ are holomorphic at z.

The function f (or the form α) is said to be holomorphic in the set $D \subset Z$ if it is holomorphic at all points of the set D.

Functions holomorphic at a point $z \in Z$ form a ring of integrity $\mathfrak{O}_z$. This ring is isomorphic to the ring of convergent power series centered on the point. Theorems 4.5 and 4.8 hold for it. It is therefore a Noether ring and is subject to the theorem on the uniqueness of the decomposition of the function $f \in \mathfrak{O}_z$ into a product of irreducible functions belonging to the ring $\mathfrak{O}_z$.

Functions that are holomorphic in some region $D \subset Z$ also form a ring of integrity; a ring formed of functions that are holomorphic in some open set $E \subset Z$ is not in general a domain of integrity.

The ensemble of rings $\mathfrak{O}_z$ for various points $z \in E$ forms a bundle usually denoted by the symbol $\mathfrak{O}(E)$ (in particular we may have $E = Z$).

The notion of analytic, sparse, and almost sparse sets is defined for the complex manifold Z in precisely the same way as for the space C^n. There is also no change in the meaning of the concepts of irreducible and locally irreducible analytic sets, of their ordinary and singular points, of the notion of germ, simple germ, and locally irreducible simple germ of an analytic set (cf. subsections 4 and 5 of §4, Chapter I).

Theorem 6.7 (Riemann's theorem on analytic continuation of holomorphic functions) holds for complex manifolds.

The definition of functions that are meromorphic in a region $D \subset Z$, or at a point $z \in Z$, is the same as that given before (cf. subsection 2, §9, Chapter II). Since Theorem 4.5 on the uniqueness of the decomposition of a function $f \in \mathfrak{O}_z$ into irreducible factors holds in a complex manifold, the ring $\mathfrak{O}$ is integrally closed in its quotient field $\mathfrak{M}_z$.

The notion of holomorphic function generalizes to that of holomorphic mapping.

DEFINITION (holomorphic mapping). Let D and D^* be open subsets of the complex manifolds Z and Z^*. A continuous mapping $T: D \longrightarrow D^*$ is called a holomorphic mapping from the manifold Z to the manifold Z^* if to every function ϕ holomorphic in some open set $D_1^* \subset D^* \subset Z^*$ there corresponds in the open set $D_1 \subset D \subset Z$ a holomorphic function $\phi \circ T$. Here $D_1 = T^{-1} D_1^*$.

If the inverse image T^{-1}: $T(D) \longrightarrow D$ of the holomorphic mapping T is itself a holomorphic mapping from the manifold Z^* to the manifold Z, then T is said to be *biholomorphic*.

If $Z^* = C^1$, the definition of holomorphic mapping reduces to that of holomorphic function on the complex manifold Z. If $Z^* = C_w^n$, where C_w^n is the space of the complex variables $w_1, \cdots, w_n$, the condition that T be holomorphic is equivalent to the requirement that the functions $w_k = w_k(z)$, $z \in Z$, $k \doteq 1, \cdots, n$ be holomorphic, these being the functions that define the mapping.

If the holomorphic mapping μ: $Z_k \longrightarrow W_n$ defines a locally regular or regular imbedding of the complex manifold Z_k in the complex manifold W_n (cf. subsection 4 of the Introduction), we call the imbedding analytically locally regular or analytically regular. If, moreover, $Z_k \subset W_n$, we shall speak of an analytically locally regularly imbedded, or analytically regularly imbedded, complex submanifold Z_k of the complex manifold W_n.

In this case the relations $(0.1_1)-(0.1_5)$ are to be written in terms of holomorphic functions of the complex coordinates of the corresponding points of the manifolds W_n and Z_k.

A complex manifold is always orientable. This follows from the fact that the (real) Jacobian of the biholomorphic mapping $\psi_j(U_i \cap U_j) \longrightarrow \psi_i(U_i \cap U_j)$ is always positive.

2. Complex analytic covering of a complex manifold.

DEFINITION. The triple $\mathfrak{W} = (W, \eta, Z)$ is called a *complex analytic covering* of the complex manifold Z if

1) W is a locally compact Hausdorff space and η a continuous non-singular and nowhere exceptional mapping of the space W on the manifold Z.

2) The manifold Z contains an analytic set A such that: a) the set $\eta^{-1}(A)$ nowhere splits the space W and b) the mapping η carries the set $W \setminus \eta^{-1}(A)$ into the open set $Z \setminus A$ in a locally homeomorphic way.

We recall that a mapping is said to be proper if the inverse image of every

compact set under the mapping is itself compact. A mapping is said to be nowhere exceptional if every point of the image under the mapping has a finite set as its inverse image (cf. subsection 2, §10).

The space $\mathfrak{W}$ is called the space of the analytic covering $\mathfrak{W}$. The analytic covering $\mathfrak{W}$ is said to be connected if its space W is connected.

If the point $w \in W$, the point $z \in Z$, $\eta(w) = z$, then z is the *fundamental* point of the point w. In this case we say that the point w lies over the point z. The mapping η is called a *projection* of the space W on the manifold Z.

An analytic set $A \subset Z$ which, with respect to the covering $\mathfrak{W}$, has the properties enumerated above is called a *critical* set of $\mathfrak{W}$. If $A_1 \subset Z$, $A_2 \subset Z$ are two critical sets of the covering $\mathfrak{W}$, then their intersection $A_1 \cap A_2$ is also a critical set of that covering.

The simplest example of an analytic covering is the *trivial* analytic covering (Z, i, Z), where i is the identity mapping. We remark that a plane covering of the second kind S_j, which we considered in §8, is a trivial analytic covering of the unit polycylinder, in the space C^n, with center at the point M_j.

We consider the domain $M_{(n)}$, lying on the circle $E\{|z| < 1\}$ in the Riemann surface of the function $w = \sqrt[n]{z}$, and the domain L, lying on the annulus $E\{1 < |z| < 2\}$ in the Riemann surface of the function $w = \operatorname{Ln} z$. It is easy to see that the triple $(M_{(n)}, \gamma, E)$, where γ is the projection $z \longrightarrow \underline{z}$, is an analytic covering, and the triple (L, γ, E) is not. Neither is the triple (L_1, γ, E) where L_1 is the image of the rectangle $\{0 < \operatorname{Re} w < \ln 2,\ 0 < \operatorname{Im} w < 3\pi\}$ under the mapping $z = e^w$ on the Riemann surface of the function $w = \operatorname{Ln} z$. We note that in each of these cases the spaces $M_{(n)}$, L, L_1 are manifolds.

Examples of many-dimensional analytic coverings, and of coverings for which the covering space is not many-dimensional, will be given later (cf. subsections 4 and 5 of this section).

Two analytic coverings $\mathfrak{W}_k = (W_k, \eta_k, Z_k)$, $k = 1, 2$, $Z_2 = \rho Z_1$, where ρ is a homeomorphic mapping of the manifold Z_1 on the manifold Z_2, are said to be mutually *equivalent* if there exists a homeomorphic mapping $\lambda\colon W_2 \longrightarrow W_1$ such that $\eta_2 = \rho \circ \eta_1 \circ \lambda$.

If $\mathfrak{W}_k = (W_k, \eta_k, Z_k)$, $k = 1, 2$ is an analytic covering of the complex manifolds Z_1 and Z_2, then the triple $\mathfrak{W}_1 \times \mathfrak{W}_2 = (W_1 \times W_2, \eta_1 \times \eta_2, Z_1 \times Z_2)$ defines an analytic covering of the manifold $Z_1 \times Z_2$. Here $\eta_1 \times \eta_2$ is a mapping of the

space $W_1 \times W_2$ on the manifold $Z_1 \times Z_2$ generated by the mapping η_k: $W_k \to Z_k$, $k = 1, 2$.

We now cite (without proof)[1] several properties of analytic coverings $\mathfrak{W} = (W, \eta, Z)$.

1) The projection η: $W \to Z$ is an open mapping. If $A \subset Z$ is a critical set of the covering $\mathfrak{W}$, the triple $(W \setminus \eta^{-1}(A), \eta, Z \setminus A)$ defines an unramified covering of the set $Z \setminus A$ without boundary points.[2]

2) Every point $w \in W$ has a finite basis of neighborhoods $\{U_\nu\}$, $\nu = 1, 2, \cdots$, such that every triple (U_ν, η, V_ν), where $V_\nu = \eta(U_\nu)$, defines an analytic covering of the complex manifold V_ν.

3) If the complex manifold Z is the union of an at most countable ensemble of compact sets, the space W is paracompact and metrizable.

4) The space W is locally linearly connected. If the set $S \subset Z$ does not split the manifold Z, the set $\eta^{-1}(S) \subset W$ does not split the space W.

5) If the domain $Z' \subset Z$, the open set $W' = \eta^{-1}(Z')$ is uniquely decomposed into a finite set of connected components W'_ν $(\nu = 1, \cdots, s)$. Every triple (W'_ν, η, Z') defines a connected analytic covering of the domain Z'.

6) If the covering $\mathfrak{W}$ is connected, the number of points $w \in W$ lying over an arbitrary point $z \in Z \setminus A$ is the same for all such points z'.

This number is called the *number of sheets of the analytic covering* $\mathfrak{W}$ and is denoted by the symbol $b = b(\mathfrak{W})$. The covering $\mathfrak{W}$ is said to be *b-sheeted.* The number of points $w \in W$ lying over an arbitrary point $z \in A$ is then $\leq b$.

DEFINITION (order of a point, point of one-sheetedness or univalence, branch point). The point $w \in W$ of the covering space W of some analytic covering $\mathfrak{W} = (W, \eta, Z)$ is said to be a *point of order* k if it has a basis of neighborhoods $\{U_\nu\}$ $(\nu = 1, 2, \cdots)$ such that all the analytic coverings $(U_\nu, \eta, \eta(U_\nu))$ are k-sheeted. The order of the point $w \in W$ will be denoted by the symbol $O(w)$. If $O(w) = 1$, then w is a *point of one-sheetedness* (or a *point of*

1) The proofs of these assertions (or an indication of where they can be found) are given in Grauert-Remmert [2]. In constructing the basic concepts of the theory of complex spaces we have in general followed this paper.

2) A mapping of a topological space onto another is said to be an open mapping if it carries every open set into an open set. The second part of the assertion is a particular case of a theorem on covering homotopy (cf. Steenrod, *The Topology of Fibre Bundles*, Vol. 14, Princeton Univ. Press, Princeton, N. J., 1951.

univalence) for the covering $\mathfrak{W}$. Otherwise w is a branch point.

If $V \subset W$ is the set of branch points and $A \subset Z$ is a critical set for the covering $\mathfrak{W}$, then $V \subset \eta^{-1}(A)$; the set V is always closed in W and nowhere splits the space W. The triple $(W \setminus \hat{V}, \eta, Z \setminus \eta(V))$, where $\hat{V} = \eta^{-1}(\eta(V))$, defines a covering on the set $Z \setminus \eta(V)$ having neither boundary points nor branch points.

It may be shown also that:

7) If $\mathfrak{W}$ is a connected covering, then for every point $z \in Z$ we have the equation $\sum_{w \in \eta^{-1}(z)} \mathrm{O}(w) = b(\mathfrak{W})$.

8) Every point $w \in \eta^{-1}(A)$, where A is a critical set of the covering $\mathfrak{W}$, at which the (complex) codimension of the set is ≥ 2 is a point of one-sheetedness for the covering $\mathfrak{W}$.

This assertion shows that for an arbitrary analytic covering $\mathfrak{W}$ with a nonempty critical (analytic) set A, the latter can always be so chosen as to have a (complex, pure) codimension equal to 1.

THEOREM 14.1 (on extension of an analytic covering). *Suppose that* a) M *is a nowhere dense analytic subset of the complex manifold* Z; b) $(W', \eta', Z \setminus M)$ *is an analytic covering of the manifold* $Z \setminus M$, *and some critical set* $A \subset Z \setminus M$ *of this covering can be analytically extended, at an arbitrary point* $\zeta \in M$, *to the space* $(Z \setminus M) \cup V_\zeta$ *where* V_ζ *is a suitable neighborhood of the point* ζ *in the manifold* Z. *Then, uniquely to within equivalence, there exists one and only one analytic covering* (W, η, Z) *such that the analytic covering* $(\eta^{-1}(Z \setminus M), \eta, Z \setminus M)$ *is equivalent to the given covering* $(w', \eta', Z \setminus M)$.

3. Normalization of an analytic set. Examples of analytic coverings. The notion of normalization of an analytic set is useful for the construction of examples of analytic coverings.

Let Z be a complex manifold. We consider initially the ensemble $P_z = \{p_z\}$ of simple germs of analytic sets for each point $z \in Z$, and the union $P(Z) = \{P_z, z \in Z\}$ of these for all $z \in Z$. In this union, of which the elements (points) are the simple germs p_z, we define a basis of open sets $\{m^*\}$. Every such set $m^* \subset P(Z)$ is an ensemble of simple germs of analytic sets belonging to some analytic set m in the manifold Z.

The union $P(Z)$ is therefore a topological space. We refer to it later as the space of simple germs of analytic sets in the complex manifold Z.

The space $P(Z)$ so defined turns out to be locally compact and locally linearly connected.

We denote by μ the projection $m^* \longrightarrow m$. This projection sets up a continuous correspondence between the elements (points) of the open set $m^* \subset P(Z)$ and the points of the analytic set $m \subset Z$. The pair (m^*, μ) will be called a *normalization* of the analytic set m in the space $P(Z)$. The following theorem is true:

THEOREM 14.2. *The analytic set m is irreducible in the complex manifold Z if and only if the set m^* is connected. The set m is locally irreducible at the point $z \in Z$ if and only if the projection μ defines a homeomorphic correspondence between the sets $m \cap V_z$ and $\mu(m \cap V_z) \subset m^*$. Here $V_z \subset Z$ is some neighborhood of the point z.*

EXAMPLES. As was shown in subsection 5, §4 of Chapter I, 1) the analytic set $m_1\{w^4 - z_1^2 z_2^2 = 0\}$ in the space C^3 of the variables w, z_1, z_2 is reducible, since it breaks down into the two irreducible analytic sets $m_1^{(1)}\{w^2 - z_1 z_2 = 0\}$ and $m_1^{(2)}\{w^2 + z_1 z_2 = 0\}$; it is connected, since at the origin of coordinates, i.e., at the point $O \in C^3$, these sets coincide. The set m_1^* in the space $P(C^3)$ is not connected; the simple germs $(m_1^{(1)})_0$ and $(m_1^{(2)})_0$ define different points. 2) The analytic set $m_2\{w^2 - z_1^2 z_2 = 0\}$ is irreducible at the origin of coordinates, i.e., the point $O \in C^3$, but is not locally irreducible there.

Therefore the projection $m_2^* \longrightarrow m_2$ is not a homeomorphism in any neighborhood of the point O. One can always find there a point $z \in m_2$ for which there are two corresponding points (two simple germs) belonging to the set m_2^*.

THEOREM 14.3. *If m is an analytic set in the manifold Z and (m^*, μ) is a normalization of it in the space $P(Z)$, then*

a) *The set m^* is a locally compact and locally connected topological space. The projection μ: $m^* \longrightarrow m$ is a continuous, non-singular, nowhere exceptional, surjective mapping.*

b) *If n is the set of singular points of the set m, then the set $\mu^{-1}(n)$ nowhere splits the space m^*. The bounded mapping $\mu|_{m \setminus n}$ is a homeomorphism.*

If the pair $('m^*, '\mu)$ (where $'m^* \subset P(Z)$; and $'\mu$: $'m^* \longrightarrow m$ is the corresponding mapping) has the properties listed in subsections a) and b) of the present theorem, then there exists a homeomorphic mapping τ: $'m^* \longrightarrow m^*$ for which $'\mu = \mu \circ \tau$.

The space m^* is called the *normalization space* of the analytic set m.

Let Z_1 and Z_2 be complex manifolds, m a pure-dimensional analytic set contained in the complex manifold $Z_1 \times Z_2$, $\gamma: m \longrightarrow Z_1$ a projection of the set m on the manifold Z_1, and (m^*, μ) a normalization of the analytic set m in the space of simple germs of the analytic set $P(Z_1 \times Z_2)$. Then we have the

THEOREM 14.4. *If the projection $\gamma: m \longrightarrow Z_1$ is surjective, then the triple (m^*, γ^*, Z_1), where $\gamma^* = \gamma \circ \mu$, defines an analytic covering on the manifold Z_1. The triple (m, γ, Z_1) defines an analytic covering on the manifold Z_1 if and only if the analytic set m is locally indecomposable.*

PROOF. If follows from the definition of the mapping γ as a projection, and from Theorem 14.3, that $\gamma^* = \gamma \circ \mu$ is a continuous, non-singular, nowhere exceptional, and surjective mapping. We show that the triple (m^*, γ^*, Z_1) satisfies the second condition in the definition of analytic covering. By the hypothesis of our theorem, the analytic set m, under the projection γ, covers the manifold Z_1. Therefore, there exists an analytic set $A \subset Z_1$ of dimension lower than that of Z_1, such that the restriction of the projection $\gamma: m \backslash \gamma^{-1}(A) \longrightarrow Z_1 \backslash A$ is locally homeomorphic. We may always assume that the set $\gamma^{-1}(A) \subset m$ contains all the exceptional points of the set m. Then, by Theorem 14.3, the set $\gamma^{*-1}(A) = \mu^{-1}(\gamma^{-1}(A))$ nowhere splits the space m^*, and the restriction of the mapping $\mu: m^* \backslash \gamma^{*-1}(A) \longrightarrow m \backslash \gamma^{-1}(A)$ is locally homeomorphic. Therefore the restriction of the mapping $\gamma^*: m^* \backslash \gamma^{*-1}(A) \longrightarrow Z_1 \backslash A$ is also locally homeomorphic.

Accordingly, the triple (m^*, γ^*, Z_1) defines an analytic covering of the manifold Z_1. If the set m is locally irreducible, the projection $\mu: m^* \longrightarrow m$ is a homeomorphism by Theorem 14.2. It follows that in this case the triple (m, γ, Z_1) defines an analytic covering of the manifold Z_1. Conversely, suppose that the triple (m, γ, Z_1) is known to define an analytic covering of the manifold Z_1 and that $A \subset Z_1$ is a critical analytic set of this covering. Since the set $\gamma^{-1}(A)$ nowhere splits the analytic set m and contains all its exceptional points, the set m is locally irreducible (cf. subsection 5, §4, Chapter I).

We remark that by Theorem 4.13 the assumptions of Theorem 14.4 will be fulfilled if the analytic set m has the same dimension as the manifold Z_1, while the manifold Z_2 is a bounded region $G \subset C^2$ and the intersection $\overline{m} \cap (Z_1 \times \partial G)$ is empty.

The assumptions of Theorem 14.4 will be fulfilled if $Z_2 = C^1_w$ and the set $m\{w = w(z)\}$ is defined by the equation

$$(m) \qquad w^b + a_1(z) w^{b-1} + \cdots + a_b(z) = 0 \qquad (3.1)$$

with coefficients $a_k(z)$ $(k = 1, \cdots, b)$ which are holomorphic on the manifold Z_1. In this case (m^*, γ^*, Z_1) is a b-sheeted analytic covering; it is connected if the polynomial (3.1) is irreducible in the manifold Z_1.

We now pass to the consideration of examples of analytic coverings.

EXAMPLE 1. We consider the polycylinder $Z\{|z_1| < r_1, \cdots, |z_n| < r_n\} \subset C^n_z$ (where all $r_n > 0$), the plane C^1_w, and the analytic set $m_b\{z \in Z, w^b - z_1 = 0\}$. Here $b \geq 1$ is an integer. Let γ be the 'natural' projection $(z, w) \to z$, where the point $(z, w) \in C^n_z \times C^1_w$, and z is the corresponding point in the space C^n_z. Then the triple $\mathfrak{B}_b = (m_b, \gamma, Z)$ defines a connected b-sheeted analytic covering. Every point $(z, w) \in m_b$, lying in a piece of the analytic surface $\{z_1 = 0, z \in Z\}$ is a branch point of the analytic covering $\mathfrak{B}_b$ of order b. All other points $(z, w) \in m_b$ are points of one-sheetedness.

EXAMPLE 2. We consider in the space $C^2_z \times C^1_w$ the analytic set $m\{z \in Z, w^b - z_1^{b_1} z_2^{b_2} = 0\}$, where Z is the bicylinder $\{|z_1| < 1, |z_2| < 1\}$, together with its normalization (m^*, μ) in the space of germs $P(C^2_z \times C^1_w)$; we set $\gamma^* = \gamma \circ \mu$. The triple $\mathfrak{B}^* = (m^*, \gamma^*, Z)$, by Theorem 14.4, defines an analytic covering of the bicylinder Z. Clearly, this covering is b-sheeted; it is connected if the greatest common divisor of the integers b_1, b_2 and b is equal to unity. In this case the set m is irreducible. Over the origin of coordinates of the space C^2_z lies a bth order branch point of the covering $\mathfrak{B}^*$. All other points of a piece of the analytic plane $\{z_k = 0, (z_1, z_2) \in Z\}$ (for $k = 1$ and $k = 2$) are covered by β_k branch points of the covering $\mathfrak{B}^*$. Here β_k is the greatest common divisor of b_k and b; the order of these branch points is equal to $\beta_k^{-1} b$.

The triple $\mathfrak{B} = (m, \gamma, Z)$ defines an analytic covering of the bicylinder Z if $\beta_1 = \beta_2 = 1$. In this case the analytic surface m is locally indecomposable at all its points, and in particular at the origin of coordinates.

In these examples the covering spaces may be replaced by plane covering regions. The latter may be constructed by replacing the elements of the surface m with the corresponding elements of a plane region of the covering z. These elements may be chosen so that the projection γ is a homeomorphic mapping.

4. Types of branch points. It is obvious that a point of one-sheetedness

$w \in W$ of the analytic covering $\mathfrak{W} = (W, \eta, Z)$ has a neighborhood homeomorphic to a Euclidean space, since it is a uniformizable point of the space W. Branch points of an analytic covering may also be uniformizable. Thus, all branch points of the analytic covering $\mathfrak{B}_b = (m_b, \gamma, Z)$ considered in Example 1 of the preceding subsection are uniformizable. We can easily show this by projecting the analytic set m_b on the space of variables $w, z_2, \cdots, z_n$.

A branch point $w_0 \in W$ is called a *torsion point* of order b for the analytic covering $\mathfrak{W} = (W, \eta, Z)$ if it has a neighborhood $V \subset W$ for which the analytic covering $(V, \eta, \eta(V))$ is equivalent to the covering $\mathfrak{B}_b$. It is obvious that in this case $O(w_0) = b$. A torsion point is a uniformizable point of the space W.

We have the

THEOREM 14.5. *The ordinary points of a critical set A of an analytic covering $\mathfrak{W} = (W, \eta, Z)$ can only be covered by torsion points.*

THEOREM 14.6. *A projection $\eta(V) \subset Z$ of the set $V \subset W$ of the branch points of an analytic covering $\mathfrak{W} = (W, \eta, Z)$ is either empty or is an analytic subset of the manifold Z with a pure codimension equal to unity.*

$\eta(V)$ is the smallest critical set of the covering $\mathfrak{W}$.

An exceptional point of a critical set A may also be covered by non-uniformizable branch points of the analytic covering $\mathfrak{W}$.

We choose a triple $\mathfrak{W} = \{m, \gamma, Z\}$ from the second example of the preceding subsection, with $b = 2$, $b_1 = b_2 = 1$. It defines an analytic covering on the bicylinder Z, since the analytic surface $m\{w^2 - z_1 z_2 = 0,\ z \in Z\}$ is locally indecomposable.

For the analytic covering $\mathfrak{W}$ the set $A\{z_1 z_2 = 0,\ z \in Z\}$ is a critical set. The origin of coordinates, where $z_1 = z_2 = 0$, is its only singular point. We show that it is covered by the non-uniformizable branch point $(0, 0, 0)$ of the space m.

We consider the space $Z_4^{(r)}$, an image of the auxiliary bicylinder $E^{(r)}\{|t_1| < r,\ |t_2| < r\}$ under the mapping $z_1 = t_1^2$, $z_2 = t_2^2$. The space $Z_4^{(r)}$ lies over the bicylinder $Z^{(r)}\{|z_1| < r^2,\ |z_2| < r^2\}$. The triple $\mathfrak{Z}_4^{(r)} = (Z_4^{(r)}, \gamma, Z^{(r)})$, where γ is the natural projection, defines a four-sheeted analytic covering of the bicylinder $Z^{(r)}$; the point $(z_1, z_2) \in Z^{(r)}$ for $z_1 z_2 \neq 0$ is covered by four points of the space $Z_4^{(r)}$, corresponding to the points (t_1, t_2), $(-t_1, t_2)$, $(-t_1, -t_2)$, $(t_1, -t_2)$ of the bicylinder $E^{(r)}$. The points $(z_1, z_2) \in Z^{(r)}$ for

$z_1 z_2 = 0$, but $|z_1|^2 + |z_2|^2 \neq 0$, are covered by two points of the space $Z_4^{(r)}$; they are branch points of order two for the covering $\mathfrak{Z}_4^{(r)}$. The point $(0, 0) \in Z^{(r)}$ is covered by one point of the space $Z_4^{(r)}$; it is a branch point of order four for the covering $\mathfrak{Z}_4^{(r)}$.

We now carry out a pairwise identification of the points of the space $Z_4^{(r)}$ corresponding to the points (t_1, t_2) and $(-t_1, -t_2)$ of the bicylinder $E^{(r)}$. As a result, we obtain from the space $Z_4^{(r)}$ the space $Z_2^{(r)}$, which is easily seen to be homeomorphic to the space m. The triple $\mathfrak{Z}_2^{(r)} = (Z_4^{(r)}, \zeta^{(r)}, Z_2^{(r)})$, where ζ is the natural projection, defines a continuous covering of the space $Z_2^{(r)}$. Every point of the space $Z_2^{(r)}$, except the origin O, is covered by two points of the space $Z_4^{(r)}$. The point $O \in Z_2^{(r)}$ is covered by the only second-order branch point of the covering.

We now show that under these conditions the point O cannot have a neighborhood in the space $Z_2^{(r)}$ which is homeomorphic to a Euclidean space. In fact, if such a neighborhood were to exist, the triple $(Z_4^{(r)} \setminus O, \zeta, Z_2^{(r)} \setminus O)$ (for sufficiently small r) would define a linearly connected continuous nonramified two-sheeted covering $\mathfrak{Y} = Z_4^{(r)} \setminus O$ of the singly-connected space $\mathfrak{X} = Z_2^{(r)} \setminus O$. Suppose that the point $\mathfrak{x} \in \mathfrak{X}$, and that the points $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathfrak{Y}$ are its inverse images under the mapping ζ. We consider the path $f \subset \mathfrak{Y}$ joining the points $\mathfrak{y}_1$ and $\mathfrak{y}_2$, together with its projection $\zeta f = \phi \subset \mathfrak{X}$. The path ϕ is a loop with a vertex at the point $\mathfrak{x}$. Since the space $\mathfrak{X}$ is simply connected, it contains loops ϕ_t, where $0 \leq t \leq 1$, depending continuously on the parameter t and such that $\phi_0 = \phi$, while ϕ_1 coincides with the point $\mathfrak{x}$. Let J be the set of all values of the parameter t in the closed interval $0 \leq t \leq 1$ for which there exist paths $f_t \subset \mathfrak{Y}$ joining the points $\mathfrak{y}_1, \mathfrak{y}_2$ and satisfying $\zeta f_t = \phi_t$. The set J must be both open in the interval (as follows from the definition of local homeomorphism) and closed (if the number $t_j \in J$; $j = 1, 2, \cdots$, $\lim_{j \to \infty} t_j = t_0$, then $f_{t_0} = \lim_{j \to \infty} f_{t_j}$). Therefore the set J coincides with the whole closed interval $0 \leq t \leq 1$. But then $1 \in J$ and there must exist a path f_1, joining the points $\mathfrak{y}_1, \mathfrak{y}_2$ and projecting into the path ϕ_1, that is, into the point $\mathfrak{x}$. This last is clearly impossible.[1]

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1) The latter part of the argument can be replaced by an application of a theorem on covering homotopy. We note that if the point O were to have a neighborhood homeomorphic to a Euclidean space, the space $Z_2^{(r)}$ would be a manifold. If it were a complex manifold, the covering $\mathfrak{Z}_2^{(r)}$ would be an analytic covering with a single branch point. But this is impossible in view of Theorem 14.6. The text considers the general case.

Thus we have shown that the point $O \in Z_2$ (and therefore also the point $(0, 0, 0) \in m$, where $m\{w^2 - z_1z_2 = 0\}$) are non-uniformizable points of their spaces. Of course, it follows from this that these points cannot be complex-uniformizable either. The space Z_2^∞, after deletion of the points for which $z_1z_2 = 0$, coincides with the domain of holomorphy of the function $\sqrt{z_1z_2}$. The point O is a non-uniformizable boundary point of this region. All other boundary points are complex-uniformizable.

§15. HOLOMORPHIC AND MEROMORPHIC FUNCTIONS ON A COMPLEX ANALYTIC COVERING. COMPLEX α-SPACES OF BEHNKE-STEIN

1. Holomorphic functions relative to an analytic covering.

DEFINITION. The function f, continuous on the open set $V \subset W$, is said to be *holomorphic* on that set relative to the analytic covering $\mathfrak{W} = (W, \eta, Z)$ if every point of one-sheetedness $w \in V$ has a one-sheeted neighborhood $U_w \subset V$ such that the function $f \circ \eta^{-1}$ is holomorphic in the region $\eta(U_w) \subset Z$.

The function f is said to be holomorphic at a point $w \in W$ relative to the analytic covering $\mathfrak{W}$ if it is holomorphic on some open set $V \subset W$.

In place of the analytic covering $\mathfrak{W}$ we may consider the analytic covering $(V, \eta, \eta(V))$.

The ensemble $\mathfrak{O}_{\mathfrak{W}}$ of all functions holomorphic on the space W relative to the analytic covering $\mathfrak{W}$ is obviously a ring.

If $f \in \mathfrak{O}_Z$, where $\mathfrak{O}_Z$ is the ring of functions holomorphic on the manifold Z, then the function $f \circ \eta \in \mathfrak{O}_{\mathfrak{W}}$. Thus the projection $\eta\colon W \to Z$ generates an isomorphic mapping $\eta^*\colon \mathfrak{O}_Z \to \mathfrak{O}_{\mathfrak{W}}$ of the ring $\mathfrak{O}_Z$ into the ring $\mathfrak{O}_{\mathfrak{W}}$; the ring $\mathfrak{O}_{\mathfrak{W}}$ is an extension of the ring $\mathfrak{O}_Z$ (which is a subring of it).

THEOREM 15.1. *A function f, continuous on the space W, is holomorphic on W relative to the analytic covering $\mathfrak{W} = (W, \eta, Z)$ if and only if the function $f \circ \eta^{-1}$ satisfies on the manifold Z the equation*

$$\omega^b + a_1(z)\omega^{b-1} + \cdots + a_b(z) = 0 \tag{3.2}$$

with coefficients $a_k(z)$ $(k = 1, \cdots, b)$ that are holomorphic on that manifold. Here $b = b(\mathfrak{W})$ is the number of sheets of the covering $\mathfrak{W}$.

PROOF. We consider the function $f \in \mathfrak{O}_{\mathfrak{W}}$. Let $A \subset Z$ be a critical set of the covering $\mathfrak{W}$, and let $\omega_1(z), \cdots, \omega_b(z)$ be the values of the function f

at the points $w \in W$ lying over some point $z \in Z \setminus A$. Then the symmetric functions

$$a_1(z) = \sum_{k=1}^{b} \omega_k(z), \quad a_2(z) = \sum_{p,\, q=1,\, p \neq q}^{b} \omega_p(z)\,\omega_q(z), \cdots$$

are holomorphic on the set $Z \setminus A$. In fact, by continuation along a closed path lying in this set, the values of the functions $\omega_k(z)$ may be permuted among themselves; but the values of the symmetric functions $a_k(z)$ are unchanged. As the point z approaches the set A these symmetric functions $a_k(z)$ remain bounded, since the function f is continuous. By Riemann's theorem on continuation of holomorphic functions (Theorem 6.7), which is valid on manifolds, it follows that the functions $a_k(z)$ are holomorphic on the entire manifold Z. It is obvious that the function f satisfies equation (3.2) with the coefficients defined in this way.

We now assume that the function f is continuous on the space W, and that the function $f \circ \eta^{-1}$ satisfies equation (3.2) on the manifold Z. Then we can show that this function is holomorphic at points $z \in Z$ not belonging to the discriminant set of equations (3.2), by applying Theorem 2.2, the implicit function theorem. We then consider the holomorphy of the function $f \circ \eta^{-1}$ at the remaining points $z \in Z \setminus A$, applying Theorem 6.7 (Riemann's theorem) on continuation of holomorphic functions.

REMARK. In other words, Theorem 15.1 establishes the fact that the ring $\mathfrak{O}_{\mathfrak{W}}$ coincides with the set of all algebraic integers f relative to the ring $\mathfrak{O}_Z$ of degree $(f \circ \eta^{-1}\colon \mathfrak{O}_Z) \leq b = b(\mathfrak{W})$. The degree of an algebraic number relative to a ring is that of the equation of lowest degree, with coefficients from the ring, satisfied by the number.

We take note of the following theorem, first proved by Osgood in a slightly less general form.[1)]

THEOREM 15.2. *Let m be a purely d-dimensional analytic subset of the region $G^n = G^d \times G^{n-d} \subset C^n$ (where $d \geq 1$), and let the projection $\gamma\colon m \to G^d$ be surjective; let (m^*, μ) be a normalization of the set m. Then if the function f^* is holomorphic in m^* relative to the analytic covering $(m^*, \gamma \circ \mu, G^d)$, there exist in the region G^d two holomorphic functions f_1 and f_2 such that the set*

1) Cf. Osgood [1], p. 116.

$n^* = \{z^* \in m^*, f_2 \circ \mu(z^*) = 0\}$ *is nowhere dense in the set* m^*, *and the equation*

$$f^* = \frac{f_1 \circ \mu(z^*)}{f_2 \circ \mu(z^*)}$$

holds on the set $m^* \setminus n^*$.

The definitions of analytic, sparse, and almost sparse sets (cf. subsection 1, §9), Theorem 6.7 (Riemann's theorem) on continuation of holomorphic functions, and the definition of holomorphic mapping (cf. subsection 1 of the preceding section) are all transferred without essential changes to the case of analytic coverings.

We content ourselves with stating the following definition.

DEFINITION (holomorphic mapping). Let $D \subset W$ and $D^* \subset W^*$ be open subsets of the spaces of the analytic coverings $\mathfrak{W} = (W, \eta, Z)$ and $\mathfrak{W}^* = (W^*, \eta^*, Z^*)$. The continuous mapping $T: D \longrightarrow D^*$ is said to be a holomorphic mapping from the covering $\mathfrak{W}$ to the covering $\mathfrak{W}^*$ if to every function ϕ holomorphic on some set $D_1^* \subset D^* \subset W^*$ there corresponds in the open set $T^{-1}(D_1^*) \subset D \subset W$ the function $\phi \circ T$ holomorphic relative to the covering $\mathfrak{W}$.

If the holomorphic mapping $T: D \longrightarrow D^*$ is such that its inverse $T^{-1}: T(D) \longrightarrow D$ is a holomorphic mapping from the covering $\mathfrak{W}^*$ to the covering $\mathfrak{W}$, the mapping T is said to be *biholomorphic*.

For sparse subsets of the spaces of analytic coverings we have the following theorem.

THEOREM 15.3. *If* $N \subset W$ *is a sparse subset of the space* W *of the analytic covering* $\mathfrak{W} = (W, \eta, Z)$, *then the set* N *nowhere splits the space* W. *The projection* $\eta(N) \subset Z$ *is a sparse subset of the manifold* Z.

The consideration of functions holomorphic relative to analytic coverings allows us to single out an important class of these coverings.

DEFINITION (analytic c-covering, or algebroidal covering). An analytic covering $\mathfrak{W} = (W, \eta, Z)$ is said to be an analytic c-covering or algebroidal covering (or is said to satisfy the c-condition) if for every point $z \in Z$ we can find a neighborhood $U_z \subset Z$ and a function $f(w)$, holomorphic on the set $\eta^{-1}(U_z)$ relative to the covering $\mathfrak{W}$, such that $[f \circ \eta^{-1}: \mathfrak{O}_{U_z}] = b(\mathfrak{W})$.

Thus, an analytic c-covering has no 'superfluous' sheets, i.e., sheets for whose points all holomorphic functions take on identical values for the

same projections; the c-condition for analytic coverings corresponds to the condition of analytic separability for plane regions of a covering.

One of the most important results in the theory of complex spaces is the following.

THEOREM 15.4 (Grauert-Remmert [2]). *Every analytic covering is an analytic c-covering.*

2. Meromorphic functions relative to an analytic covering. The definitions already given for a meromorphic function and its polar set (cf. subsection 2, §9, Chapter II) go over without change to the case of analytic coverings.

Riemann's theorem on continuation of holomorphic functions yields the following theorem on meromorphic functions.

THEOREM 15.5. *A function f, holomorphic at the points of the set $W \setminus N$ relative to the analytic covering $\mathfrak{W} = (W, \eta, Z)$, where N is some sparse subset of the space W, can be extended to the whole space W as a meromorphic function if and only if for every point $w_0 \in N$ we can find a connected neighborhood $U_{w_0} \subset W$ and a function $q \not\equiv 0$ holomorphic in that neighborhood and such that the function fq is bounded on the set $U_{w_0} \setminus N$.*

In a sense, the following theorem is the converse of Theorem 15.5.

THEOREM 15.6. *Let the function f be meromorphic in the space W relative to the covering $\mathfrak{W} = (W, \eta, Z)$ and let $N \subset W$ be its polar set. Then every point $z_0 \in \eta(W)$ has a connected neighborhood $V_{z_0} \subset Z$, and there exists in it a holomorphic function $r(z) \not\equiv 0$ such that the function $(r \circ \eta) f$ can be extended analytically as a holomorphic function on the whole set $\eta^{-1}(V_{z_0})$.*

PROOF. By hypothesis, for every point $w_0 = \eta^{-1}(z_0) \in W$ there exists a neighborhood $U_{w_0} \subset W$, and a function $q \not\equiv 0$ in it, such that the function qf is bounded on the set $U_{w_0} \setminus N$. We choose the neighborhood U_{w_0} in such a way that the triple $\mathfrak{N} = (U_{w_0}, \eta, V)$, where $V = \eta(U_{w_0})$, defines an analytic covering. Then, by Theorem 15.1 the function $q \circ \eta^{-1}$ satisfies in the set V an equation of the form

$$\omega^s + a_1(z)\,\omega^{s-1} + \cdots + a_s(z) = 0 \tag{3.3}$$

with coefficients holomorphic on that set.

We consider in the neighborhood U_{w_0} the function $g(w) = \dfrac{a_s \circ \eta}{q}$; clearly this function is holomorphic inside the set $\{w \in U_{w_0}, q(w) = 0\}$. By Theorem 15.1

it can be extended to the whole set U_{w0} as a holomorphic function, since the function $g \circ \eta^{-1}$ satisfies in the set V the equation

$$\omega^s + a_{s-1}(z)\,\omega^{s-1} + a_{s-2}(z)\,a_s(z)\,\omega^{s-2} + \cdots$$
$$\cdots + a_1(z)\,a_s^{s-2}(z)\,\omega + a_s^{s-1}(z) = 0. \qquad (3.4)$$

It now follows from the relation $\left[\frac{a_s \circ \eta}{q}\right]$ $(q \cdot f) = (a_s \circ \eta) f$ that the set $V = \eta(U_{w0})$ and the function $r(z) = a_s(z)$ have the properties required in Theorem 15.6.

We may now prove an analogue of Theorem 15.1 for meromorphic functions.

THEOREM 15.7. *We consider an analytic covering* $\mathfrak{W} = (W, \eta, Z)$ *and a sparse set* $N \subset W$. *A function* f, *continuous on the set* $W \setminus N$, *is meromorphic in the space* W *relative to the covering* $\mathfrak{W}$ *and has the polar set* $Q \subset N$ *if and only if the function* $f \circ \eta^{-1}$ *satisfies on the manifold* Z *an equation*

$$\omega^b + a_1(z)\,\omega^{b-1} + \cdots + a_b(z) = 0 \qquad (3.5)$$

with coefficients $a_k(z)$ $(k = 1, \cdots, b)$ *meromorphic on the manifold* Z *and holomorphic in the set* $Z \setminus \eta(N)$. *Here* $b = b(\mathfrak{W})$ *is the number of sheets of the covering* $\mathfrak{W}$.

REMARK. We denote by K_Z and $K_{\mathfrak{W}}$ the rings of functions meromorphic on the manifold Z and on the space W relative to the covering $\mathfrak{W}$. Theorem 15.7 asserts that a function f, continuous on the set $W \setminus N$, where $N \subset W$ is a given sparse set, belongs to $K_{\mathfrak{W}}$ and has the polar set $Q \subset N$ if and only if the function $f \circ \eta^{-1}$ is an algebraic integer for the ring K_Z with degree $(f \circ \eta^{-1}\colon K_Z) = b \leq b(\mathfrak{W})$.

PROOF. We suppose to begin with that there exists a function $f' \subset K_{\mathfrak{W}}$ with the polar set $Q \subset N$ for which the restriction $f' \mid W \setminus N = f \mid W \setminus N$ holds. Then the restriction $f \mid W \setminus N$ is holomorphic; by an argument identical to that used in proving Theorem 15.1, we find that the function $f \circ \eta^{-1} \mid Z \setminus \eta(N)$ satisfies an equation of the form (3.5) with coefficients $a_k(z)$ $(k = 1, \cdots, b)$ holomorphic on the set $Z \setminus \eta(N)$. It remains to show that these coefficients can be extended as meromorphic functions. Let z_0 be an arbitrary point of the set $\eta(N)$. By Theorem 15.6 this point admits a connected neighborhood V_{z0} and there exists in this neighborhood a holomorphic function $r(z) \not\equiv 0$ such that the function $f^* = (r \circ \eta) f'$ can be extended as a holomorphic function to the whole

set $\eta^{-1}(V_{z_0})$. The function $\omega = f^* \circ \eta^{-1}$ satisfies in the region V_{z_0} the equation

$$\omega^b + a_1(z)\, r(z)\, \omega^{b-1} + \cdots + a_{b-1}(r(z))^{b-1}\, \omega + a_b(z)\,(r(z))^b = 0. \qquad (3.6)$$

Considering the coefficients of this equation as symmetric functions of its roots, we easily see that they may be extended as holomorphic functions to the whole region V_{z_0}. It follows that the coefficients in equation (3.5) may be extended, in the region V_{z_0} and accordingly to the whole of the manifold Z, as meromorphic functions.

We now postulate that the function f satisfies equation (3.5) with coefficients $a_k(z) \in K_z$ $(k = 1, \cdots, b)$. In this case there exists a nowhere dense analytic set $S \subset Z$ such that the functions $a_k(z) \mid Z \setminus S$ are holomorphic. Then by Theorem 15.1 the function $f \circ \eta^{-1}$, which is continuous on the set $W \setminus N$, can be extended analytically to the set $W \setminus (N \cup \eta^{-1}(S))$ as a holomorphic function $\hat{f}$. This latter function coincides, inside the set $N \cup \eta^{-1}(S)$, with the given function f. By our assumption, to every point $w_0 \in \eta^{-1}(S)$ there corresponds a connected neighborhood $V \subset Z$ of the point $\eta(w_0) \in S$ and a function $r(z)$, holomorphic in this neighborhood, such that all the functions $a_k^*(z) = a_k(z)\, r(z)$ $(k = 1, \cdots, b)$ can be holomorphically extended to the region V. Then the function $(r \circ \eta)\hat{f} \mid \eta^{-1}(V \setminus S)$ will remain bounded near the point w_0 since the function $\omega = [(r \circ \eta)\hat{f}] \circ \eta^{-1} \mid V \setminus S$ satisfies the equation

$$\omega^b + \sum_{k=1}^{b} a_k^*(z)\,(r(z))^k\, \omega^{b-k} = 0. \qquad (3.7)$$

It follows that the function $\hat{f}$, and therefore also the function f, can be analytically extended as a meromorphic function to the whole space W.

3. Bundle of rings of germs of holomorphic functions relative to an analytic covering. In subsection 3, §4, Chapter I we considered bundles of germs of holomorphic functions in a region $D \subset C^n$ and on an open set $B \subset C^n$. In the same way, we define a bundle $\mathfrak{O}(\mathfrak{W}) = \{\mathfrak{O}_w,\ w \in W\}$. Here $\mathfrak{O}_w$ is the ring of germs of functions holomorphic at the point $w \in W$ relative to the covering $\mathfrak{W} = (W, \eta, Z)$. It follows from Theorem 15.1 that every germ $f_w \in \mathfrak{O}_w$ is an algebraic integer, of degree less than or equal to $b(\mathfrak{W})$, in the ring $\mathfrak{O}_{\eta(w)} \in \mathfrak{O}(Z)$; thus, the ring $\mathfrak{O}_w$ is a subring of degree $b(\mathfrak{W})$ in the ring $\mathfrak{O}_{\eta(w)} \in \mathfrak{O}(Z)$. Here $\mathfrak{O}(Z)$ is the bundle of rings of germs of holomorphic functions on

the manifold Z.

For the rings $\mathfrak{O}_w$ that make up the bundle $\mathfrak{O}(\mathfrak{W})$ we have

THEOREM 15.8. *The ring* $\mathfrak{O}_w$ *is a Noether ring of integrity; it is integrally closed (in its quotient field) and is a finite* $\mathfrak{O}_{\eta(w)}$*-module.*[1] *Here* $\eta(w) \in Z$.

PROOF. We show first of all that the ring $\mathfrak{O}_w$ is a domain of integrity, that is, contains no divisors of zero. Suppose that $f_w, g_w \in \mathfrak{O}_w$ and $f_w g_w = 0 \in \mathfrak{O}_w$. In view of our assumption, there exist in some connected neighborhood $U_w \subset W$ holomorphic functions $\hat{f}, \hat{g} \in \mathfrak{O}_{U_w}$, representing germs f_w and g_w, and $\hat{f} \cdot \hat{g} = 0 \mid U_w$. The projection η maps the set $U_w \setminus N$ (where N is some set nowhere splitting the region U_w) locally homeomorphically into the manifold Z. From the definition of a function holomorphic relative to an analytic covering $\mathfrak{W}$, it follows that either $\hat{f} \mid U_w \setminus N = 0$, or $\hat{g} \mid U_w \setminus N = 0$. By continuity, either $\hat{f} \mid U_w = 0$ or $\hat{g} \mid U_w = 0$. Thus at least one of the germs f_w, g_w is zero.

We now show that the ring $\mathfrak{O}_w$ is integrally closed in its quotient field. Suppose that $h_w = f_w / g_w$, that $f_w, g_w \in \mathfrak{O}_w$, $g_w \neq 0$, and that

$$h_w^r + c_1^{(w)} h_w^{r-1} + \cdots + c_r^{(w)} = 0, \tag{3.8}$$

where $c_k^{(w)} \in \mathfrak{O}_w$ $(k = 1, \cdots, r)$. By our assumption, there exist, in some connected neighborhood $U_w \subset W$, holomorphic functions $\hat{f}, \hat{g}, \hat{c}_k$ $(k = 1, \cdots, r)$ representing germs $f_w, g_w, c_k^{(w)}$ $(k = 1, \cdots, r)$ and $\hat{g} \not\equiv 0$. The element h_w of the quotient field of the ring $\mathfrak{O}_w$ represents in the region U_w the meromorphic function $\hat{f} / \hat{g} = \hat{h}$. This function is holomorphic in the region U_w outside the analytic set $\{w' \in U_w, \hat{g}(w') = 0\}$. It remains bounded near the points of this set since it satisfies the equation

$$\hat{h}^r + \hat{c}_1 \hat{h}^{r-1} + \cdots + \hat{c}_r = 0. \tag{3.9}$$

It follows from Riemann's theorem on analytic extension of holomorphic functions

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1) A ring $\mathfrak{O}$ is said to be integrally closed (in its quotient field) if $h = f/g$, where $f, g \in \mathfrak{O}$, satisfying the equation $h^r + c_1 h^{r-1} + \cdots + c_r = 0$ with coefficients $c_k \in \mathfrak{O}$ $(k = 1, \cdots, r)$, also belongs to the ring $\mathfrak{O}$.

An abelian group $\mathfrak{R}\{f\}$ is called a finite $\mathfrak{O}$-module, where $\mathfrak{O}$ is a commutative ring with unit, or in other words a module with a finite basis $(g_1, \cdots, g_p)$, if every $f \in \mathfrak{R}$ can be represented in the form $f = \alpha_1 g_1 + \cdots + \alpha_p g_p$. Here all $\alpha_k \in \mathfrak{O}$, and all $g_k \in \mathfrak{R}$ $(k = 1, \cdots, p)$.

that the function $\hat{h} \mid U_w - \{w' \in U_w, \hat{g}(w') = 0\}$ can be extended as a holomorphic function to the whole region U_w. Thus, $\hat{h} \in \mathfrak{O}_w$, $h_w \in \mathfrak{O}_{U_w}$.

We know (cf. subsection 1 of the preceding section) that $\mathfrak{O}_{\eta(w)}$ is an integrally closed Noether ring. We know from Theorem 15.1 that every element of its subring $\mathfrak{O}_w$ (which is a domain of integrity) satisfies an equation of the form (3.2). It follows that the ring $\mathfrak{O}_w$ is a finite $\mathfrak{O}_{\eta(w)}$-module and is a Noether ring.

In conclusion we remark that for the ring $\mathfrak{O}_w$, generally speaking, we cannot have Theorem 4.5 on the uniqueness of decomposition of a holomorphic function into a product of irreducible holomorphic factors. Let us consider, for example, the analytic covering $\mathfrak{W} = (m, \gamma, Z)$, where $Z \subset C_z^2$ is the bicylinder $\{|z_1| < 1, |z_2| < 1\}$, m is the analytic surface $\{w^2 - z_1 z_2 = 0, z \in Z\} \subset C_z^2 \times C_w^1$, and γ is the projection $(w, z) \to z$. We denote by f_m the trace (restriction) of the function $f(w, z_1, z_2)$ on the surface m. It is obvious from the defining equation of this surface that $(w)_m^2 = (w)_m (w)_m = (z_1)_m \cdot (z_2)_m$. The functions $(w)_m$, $(z_1)_m$, $(z_2)_m$ are holomorphic relative to the analytic covering $\mathfrak{W}$ at the origin of coordinates, the point $O \in m$, and the ratios $(w)_m / (z_k)_m$ and $(z_k)_m / (w)_m$ $(k = 1, 2)$ do not belong to the ring $\mathfrak{O}_m$.

It is clear from what we have said that the germ of a holomorphic function belonging to the ring $\mathfrak{O}_0 \in \mathfrak{O}(\mathfrak{W})$ and represented by the function $(w^2)_m$, can be decomposed into irreducible factors in two non-equivalent ways.

4. Complex α-spaces are defined analogously to complex manifolds with the aid of a complete atlas of holomorphically compatible α-charts.

DEFINITION (α-chart). An α-chart on the Hausdorff space R is a triple $(U, \psi, \mathfrak{W})$, where U is a non-empty open subset of the space R, $\mathfrak{W} = (W, \eta, G)$ is an analytic covering of the region $G \subset C^n$, ψ is a homeomorphic mapping of the set U into the space W. The number n is called the (complex) dimension of the α-chart.

DEFINITION (holomorphically compatible α-charts). The α-charts $(U_1, \psi_1, \mathfrak{W}_1)$ and $(U_2, \psi_2, \mathfrak{W}_2)$ on the Hausdorff space R, where $\mathfrak{W}_1 = (W_1, \eta_1, G_1)$ and $\mathfrak{W}_2 = (W_2, \eta_2, G_2)$ are corresponding analytic coverings, are said to be holomorphically compatible if either the intersection $U_1 \cap U_2$ is empty or the mapping $\psi_2 \circ \psi_1^{-1}$: $\psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$ of the space W_1 on the space W_2 is a biholomorphic mapping from the analytic covering $\mathfrak{W}_1$ to the analytic covering $\mathfrak{W}_2$.

DEFINITION (α-atlas). An ensemble of holomorphically compatible α-charts $(U_j, \psi_j, \mathfrak{W}_j)$, $j \in J$, where J is some set of indices, on the Hausdorff space R is called an α-atlas if $\bigcup_{j \in J} U_j = R$. An α-atlas is called a *complete* or *structural* α-atlas on the space R if there exists no α-chart on the space which is holomorphically compatible with the α-charts of the atlas and does not belong to the atlas.

It is easy to see that every α-atlas on a Hausdorff space can be completed to a complete α-atlas.

DEFINITION (complex α-space). A Hausdorff space R, on which a complete α-atlas is defined, is called a *complex α-space.*

It follows from Theorem 15.4 that for every α-chart $(U, \psi, \mathfrak{W})$, making up the α-atlas of the complex α-space R, an analytic covering $\mathfrak{W}$ is an analytic c-covering. Whenever we wish to draw attention to this fact, we shall call the complex α-space a *complex α_c-space.*

A Hausdorff space R on which an α-atlas is given is homeomorphic in the neighborhood of each of its points to the space of some analytic covering. Therefore such a space R is always locally compact and locally linearly connected; it is decomposed in a natural way into components R_κ, where $\kappa \in K$ and K is a set of indices. Every space R_κ has a complex dimension $d(R_\kappa)$ defined as the general dimension of its α-chart. The complex dimension of the whole space is defined as $d(R) = \sup_{\kappa \in K} d(R_\kappa)$ and may be infinite. If $d(R_\kappa) = d$ for all κ, the space R is said to be purely d-dimensional; in this case we write $R = R^d$.

The space W of an analytic covering $\mathfrak{W} = (W, \eta, Z)$, in particular every complex manifold Z, is always a complex α-space. In order to see this, one must consider for the analytic covering $\mathfrak{M} = (M, \zeta, \Gamma)$ the covering of the manifold Γ by open sets $U_j (j \in J,\ \bigcup_{j \in J} U_j = \Gamma)$ which under the biholomorphic mappings ψ_j go over into a region of the space C^{n_j}. Then we may take as α-charts the triples $(\zeta^{-1}(U_j), i, \mathfrak{M}_j)$, where i is the identity mapping, and $\mathfrak{M}_j = (\zeta^{-1}(U_j), \psi_j \circ \zeta, \psi_j(U_j))$ is the corresponding analytic covering.

Let r be a point of the complex α-space R. If the α-atlas of this space contains the α-chart $(U, \psi, \mathfrak{W})$, $r \in U$, with the trivial covering $\mathfrak{W}$, then r is a complex-uniformizable point of the space R. A complex manifold is an α-space consisting of complex-uniformizable points.

It is evident that complex-uniformizable points are always uniformizable,

or in other words have a neighborhood homeomorphic to an entire Euclidean space of the corresponding dimension. Up to now no examples are known of uniformizable points of complex α-spaces that are not complex-uniformizable.

An α-atlas of a complex α-space R permits us to define quite naturally the α-atlas of every non-empty open subset $R' \subset R$. As a result, the space R' becomes a complex α-space. The α-atlas of the topological product of α-spaces is defined in terms of the α-atlases of the spaces entering the product. As a result the product is an α-space.

DEFINITION (holomorphic function). The function f is said to be holomorphic in some open subset V of the complex α-space R if at every point $r \in V$ one can find in the α-atlas of the space an α-chart $(U, \psi, \mathfrak{W})$, $r \in U \subset V$, such that the function $f \circ \psi^{-1}$ is holomorphic on the set $\psi(U)$ relative to the covering $\mathfrak{W}$.

From the holomorphic compatibility of α-charts we derive the following assertion: if $(U', \psi', \mathfrak{W}')$ is some other α-chart of the atlas of the space R, and $r \in U' \subset V$, then in our case the function $f \circ \psi'^{-1}$ will be holomorphic relative to the covering $\mathfrak{W}^{-1}$. Thus, thanks to the holomorphic compatibility of α-charts making up the atlas of a space R, the definition of a holomorphic function does not depend on the choice of the α-chart on which the definition is based.

The function f is said to be holomorphic at the point $r \in R$ if it is holomorphic in some open set $U \subset R$, $r \in U$.

The above definition of analytic set is extensible without change to the case of a complex α-space. One can show that a nowhere dense analytic set nowhere splits a complex α-space. It is easy to see that for complex α-spaces Riemann's theorem (Theorem 6.7) on the extension of holomorphic functions preserves its validity.

The above definition of holomorphic mapping for analytic coverings is valid for complex α-spaces.

The ensemble of holomorphic functions $f | V$, where V is some open subset of the complex α-space R, constitutes a ring which we will denote by the symbol $\mathfrak{O}_V^{(\alpha)}$. Just as in all the preceding instances, we define $\mathfrak{O}_r^{(\alpha)}$ to be the ring of germs of holomorphic functions at the point r of the complex α-space R. The union $\mathfrak{O}^{(\alpha)}(R) = \{\mathfrak{O}_r^{(\alpha)}, r \in R\}$ defines in a natural manner a bundle which is called an *α-structural bundle* of rings of germs of holomorphic functions of the space.

When we wish to emphasize the fact that a complex α-space is an α_c-space, its α-structural bundle will be called an α_c-structural bundle.

It follows from Theorem 15.8 that every ring $\mathfrak{O}_r^{(\alpha)}$ is an integrally closed Noether ring of integrity. However, for this ring we do not in general have Theorem 4.5 on the uniqueness of the decomposition of a holomorphic function into a product of holomorphic irreducible and mutually non-equivalent factors.

We denote by $\mathfrak{M}_r^{(\alpha)}$ the quotient field [1] of the ring $\mathfrak{O}_r^{(\alpha)}$. We will call the field $\mathfrak{M}_r^{(\alpha)}$ the field of germs of meromorphic functions at the point r. The union $\mathfrak{M}^{(\alpha)}(R) = \{\mathfrak{M}_{(r)}^{\alpha}, r \in R\}$ constitutes in a natural way a bundle of fields of germs of meromorphic functions of the α-space R. A meromorphic function in some open set $V \subset R$ is defined as the section of the bundle $\mathfrak{M}^{(\alpha)}(R)$ over the set V (cf. the definition of section of a bundle given in subsection 10 of the Introduction). We note that in the particular α-spaces now being considered (plane surfaces of coverings, manifolds, analytic coverings) the definition reduces to the one given earlier.

The bundle $\mathfrak{O}^{(\alpha)}(R)$ is a subbundle of the bundle $\mathfrak{M}^{(\alpha)}(R)$.

In conclusion we remark that complex α-spaces were first studied by Behnke and Stein [3] in 1951.

§16. COMPLEX β-SPACES OF SERRE

1. Spaces with ring structure. [2]

DEFINITION. A topological space R is called a *space with a ring structure* if there has been singled out, from the bundle $\mathfrak{G}(R)$ of rings of germs of continuous functions (taking on complex values), a subbundle $\mathfrak{A}(R) = \mathfrak{A}_r$, $r \in R$, of the bundle $\mathfrak{G}(R)$. The subbundle $\mathfrak{A}(R)$ consists of rings $\mathfrak{A}_r$ that are subrings of the rings $\mathfrak{G}_r$. Every ring $\mathfrak{A}_r$ contains the ring Γ of constant functions. The bundle $\mathfrak{A}(R)$ is called the structural bundle of the space R.

The space R on which a ring structure has been defined with the aid of the bundle $\mathfrak{A}(R)$ is denoted by the symbol $(R, \mathfrak{A})$.

As an example of a space with a ring structure we may take the complex α-space R. Its structural bundle is the bundle $\mathfrak{O}^{(\alpha)}(R)$ of rings of germs of holomorphic functions. We will denote it by the symbol $(R, O^{(\alpha)}(R))$.

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1) The set $\mathfrak{M}_r^{(\alpha)}$ is a field since $\mathfrak{O}_r^{(\alpha)}$ is a ring of integrity.

2) Cf. H. Cartan [5, 7]; Grauert-Remmert [2], pp. 274 ff.; Serre [1].

DEFINITION (morphic function). The morphic functions on the open set $U \subset (R, \mathfrak{A})$ are the sections of the bundle $\mathfrak{A}(R)$ over this set. A function is called morphic at the point $r \in R$ if it is morphic on some open set $U \subset R$, $r \in U$. The germs of continuous functions $f_r \in \mathfrak{A}_r$ are called germs of the morphic functions at the point r. If the germ $f_r \in \mathfrak{A}_r$ belongs to the section f of the bundle $\mathfrak{A}(R)$ over the set U, one says that the function f represents the germ f_r on the set U.

The ensemble of functions morphic on some open set $U \subset (R, \mathfrak{A})$ constitutes a commutative ring with a unit, which we denote by the symbol $\mathfrak{A}_U$.

We denote by $\mathfrak{A}(U)$ the restriction of the bundle $\mathfrak{A}(R)$ on this set U; by $(U, \mathfrak{A}(U))$ the space U in which a topology is induced by the topology of the containing space R, and a ring structure is introduced with the aid of the bundle $\mathfrak{A}(U)$.

We denote by $\mathfrak{M}_r$ the ring consisting of ratios of elements of the ring $\mathfrak{A}_r$, by $\mathfrak{M}(R) = \{\mathfrak{M}_r, r \in R\}$ the bundle formed in the natural way from these rings. It is clear that the bundle $\mathfrak{A}(R)$ is a subbundle of the bundle $\mathfrak{M}(R)$. The *rhomorphic functions* on an open set $U \subset (R, \mathfrak{A})$ are the sections of the bundle $\mathfrak{M}(R)$ over that set. A function is said to be *rhomorphic at the point* $r \in R$, if it is rhomorphic on some open set $U \subset R$, $r \in U$. The elements of the ring $\mathfrak{M}_r$ are called germs of rhomorphic functions at the point $r \in R$, and the bundle $\mathfrak{M}(R)$ is the bundle of rings of germs of rhomorphic functions for the space $(R, \mathfrak{A})$.

Let $(R, \mathfrak{A})$ and $(S, \mathfrak{B})$ be spaces with ring structures, let $\mathfrak{A}(R) = \{\mathfrak{a}_r, r \in R\}$ and $\mathfrak{B}(S) = \{\mathfrak{b}_s, s \in S\}$ be their structural bundles of rings of germs of morphic functions, and let $D \subset (R, \mathfrak{A})$ and $E \subset (S, \mathfrak{B})$ be open subsets of these spaces. A continuous mapping $T\colon D \longrightarrow E$ is said to be a *morphic* mapping from the space $(R, \mathfrak{A})$ to the space $(S, \mathfrak{B})$ if to every function ϕ, morphic on some open set $E_1 \subset E \subset (S, \mathfrak{B})$, there corresponds on the open set $T^{-1}(E_1) \subset D \subset (R, \mathfrak{A})$ the morphic function $\phi \circ T$.

A morphic mapping $T\colon D \longrightarrow E$ is characterized by the fact that the correspondence $\phi_s \longrightarrow \phi_s \circ T$ between the functional germs $\phi_s \in \mathfrak{b}_s$ and $\phi_s \circ T \in \mathfrak{a}_r$ defines a homomorphic mapping $T_r^*\colon \mathfrak{b}_s \longrightarrow \mathfrak{a}_r$ of the ring $\mathfrak{b}_s$ into the ring $\mathfrak{a}_r$. Here $s = Tr$.

The morphic mapping $T\colon D \longrightarrow E$ is said to be *bimorphic* if the inverse mapping $T^{-1}\colon T(D) \longrightarrow D$ is morphic.

A closed set $m \subset (R, \mathfrak{A})$ is called an $\mathfrak{A}$-*set* in the space $(R, \mathfrak{A})$ if every point $r \in (R, \mathfrak{A})$ has a neighborhood $U_r \subset (R, \mathfrak{A})$ such that the set $m \cap U_r$ coincides with the set of common zeros of some finite set of functions that are morphic on the neighborhood U_r.

Analytic sets are $\mathfrak{O}^{(\alpha)}$-sets for complex α-spaces.

To every $\mathfrak{A}$-set m of the space $(R, \mathfrak{A})$ corresponds a bundle of ideals $I(m) = \{I_r, r \in R\}$. Here I_r is a proper ideal (or ideal, for short) of the set m at the point r. It consists of germs $f_r \in \mathfrak{A}_r$ for which there exist representative morphic functions f vanishing on the sets $m \cap U_{f_r}$. Here U_{f_r} is a neighborhood of the point $r \in R$; the choice of the neighborhood depends in general on the germ f_r.

An $\mathfrak{A}$-set $m \subset (R, \mathfrak{A})$ can be thought of as a topological space with a topology induced by that of the containing space $(R, \mathfrak{A})$. We define a ring structure on the space m. Let $U \subset (R, \mathfrak{A})$ and $U' = m \cap U$ be open subsets of the spaces $(R, \mathfrak{A})$ and m, let f' be a continuous function on the set U' which is the trace $f|U'$ of some morphic function f on U. We consider the set $\{f'\}$ of such functions for the open subsets $U \subset (R, \mathfrak{A})$, their germs f_r' and the rings $\mathfrak{A}_r'$ formed by these germs at the points $r \in m$, and finally the bundle $\mathfrak{A}'(m) = \{\mathfrak{A}_r', r \in m\}$ formed in the natural way from the rings $\mathfrak{A}_r'$. The bundle $\mathfrak{A}'(m)$ defines the *induced ring structure over the subspace* $m \subset (R, \mathfrak{A})$. It is easy to see that the imbedding $i: (m, \mathfrak{A}') \to (R, \mathfrak{A})$ is a morphic mapping. The space $(m, \mathfrak{A}')$ is a subspace with ring structure of the space $(R, \mathfrak{A})$.

We consider the topological product $(R, \mathfrak{A}) \times (S, \mathfrak{B})$ of two spaces with ring structure. Let $U \subset (R, \mathfrak{A})$ and $V \subset (S, \mathfrak{B})$ be open subsets of these spaces. We now construct the set of those continuous functions $f(r, s)$ on the set $U \times V$ (where $r \in U$, $s \in V$) for which the functions $f(r_0, s)$ and $f(r, s_0)$ for fixed $r_0 \in U$ and $s_0 \in V$ are morphic on the sets V and U with respect to the corresponding structural bundles. These functions form a ring, and the sets $U \times V$, where $U \subset (R, \mathfrak{A})$, $V \subset (S, \mathfrak{B})$, form a basis of open sets. It is therefore easy to show with the aid of the functions $f(r, s)$ that a ring structure is defined on the space $(R, \mathfrak{A}) \times (S, \mathfrak{B})$.

The space $(R, \mathfrak{A})$ belongs to the *class F of spaces with ring structure* if its morphic functions and $\mathfrak{A}$-sets satisfy a theorem like Theorem 6.7, Riemann's theorem on analytic extension of holomorphic functions. In this case, if $U \subset (R, \mathfrak{A})$ is an open set and $m \subset U$ an $\mathfrak{A}$-set nowhere dense in U, then

every morphic and bounded function $f | U \setminus m$ can be extended to a function $\tilde{f} | U$, morphic on the whole of the set U.

It is easy to see that in the case under discussion this assertion can be widened to include morphic mappings: a continuous mapping $T: (R, \mathfrak{A}) \longrightarrow (S, \mathfrak{B})$, where $(R, \mathfrak{A})$ is a space of class F, which is morphic outside some (nowhere dense) $\mathfrak{A}$-set $m \subset (R, \mathfrak{A})$ is also morphic on the whole space $(R, \mathfrak{A})$.

In conclusion, we note that complex α-spaces belong to the class F.

2. Complex β-spaces. For every open set $B \subset C_z^n$ we can define a bundle $\mathfrak{O}(B) = \{\mathfrak{O}_z, z \in B\}$ consisting of rings of germs of functions holomorphic at the points $z \in B$. As we saw in the preceding subsection, for every analytic set $m \subset B$ we may define, starting from the bundle $\mathfrak{O}(B)$, the structural bundle $\mathfrak{O}(m)$. The space m then becomes a subspace $(m, \mathfrak{O}(m))$ of the space $(B, \mathfrak{O}(B))$ with the induced ring structure $\mathfrak{O}(m)$.

DEFINITION (complex β-space). A Hausdorff space $(R, \mathfrak{A}(R))$ with ring structure is called a complex β-space if every point $r \in R$ has a neighborhood U_r such that the space $(U_r, \mathfrak{A}(U_r))$ with ring structure is bimorphically mapped on the space $(m, \mathfrak{O}(m))$ with ring structure. Here m is some analytic subset of an open set $B \subset C^n$. $\mathfrak{O}(m)$ is the bundle defining the induced ring structure on the space m, starting with the bundle $\mathfrak{O}(B)$.

In this case the bundle $\mathfrak{A}(R)$ is called the *β-structural bundle on the space R* and is denoted by the symbol $\mathfrak{O}^{(\beta)}(R)$.

In complex β-spaces the morphic and rhomorphic functions are usually called holomorphic and meromorphic functions, morphic and bimorphic mappings are called holomorphic and biholomorphic mappings, and $\mathfrak{A}$-sets are called analytic sets. The concepts of sparse and almost sparse sets carry their usual sense. When we are looking simultaneously at a number of complex structures on a topological space we use the notation: α-holomorphic function, β-holomorphic function, etc.

Examples of complex β-spaces are: complex manifolds, where the structural bundle is the bundle of rings of germs of holomorphic functions; analytic subsets of these manifolds with the ring structure induced by the structural bundle of functions holomorphic on the whole manifold.

We consider a locally indecomposable analytic set $m \subset Z\{| z_k | < r_k, k = 1, \cdots, n\}$ with the pure complex dimension d. We assume that the projection $\gamma: m \longrightarrow Z^d \{| z_k | < r_k, k = 1, \cdots, d < n\}$ is surjective. Then in view of

Theorem 14.4 the triple (m, γ, Z^d) defines an analytic covering; we use it to define on the space m an α-structure and we construct the bundle $\mathfrak{O}^{(\alpha)}(m)$ of rings of germs of holomorphic functions. On the other hand, we can construct on the space m a bundle $\mathfrak{O}^{(\beta)}(m)$ defining a β-structure. This bundle $\mathfrak{O}^{(\beta)}(m)$ is induced on the space m by the bundle $\mathfrak{O}(Z)$ of rings of germs of functions holomorphic on the polycylinder $Z \subset C^n$. It is easy to see that $\mathfrak{O}^{(\beta)}(m) \subset \mathfrak{O}^{(\alpha)}(m)$ and, in general, $\mathfrak{O}^{(\beta)}(m) \neq \mathfrak{O}^{(\alpha)}(m)$.

Consider, for example, the locally indecomposable analytic set $m\{z_2^{p_1} - z_1^{p_2} = 0, |z_1| < 1\} \subset C_z^2$, where $p_1 > 1$, $p_2 > 1$ are relatively prime whole numbers. On this set we define an α-structural bundle $\mathfrak{O}^{(\alpha)}(m)$ with the aid of the covering $\{m, \gamma: (z_1, z_2) \to z_1, |z_1| < 1\}$, and a β-structural bundle $\mathfrak{O}^{(\beta)}(m)$ by starting with the bundle $\mathfrak{O}(Z)$ of rings of germs of functions holomorphic on the polycylinder $Z\{|z_1| < 1, |z_2| < 1\}$. Then at all points $\mu \in m$, except at the point O (the origin of coordinates), $\mathfrak{O}_\mu^{(\alpha)} = \mathfrak{O}_\mu^{(\beta)}$. On the other hand, generally speaking, $\mathfrak{O}_0^{(\alpha)} \neq \mathfrak{O}_0^{(\beta)}$, $\mathfrak{O}_0^{(\beta)} \subset \mathfrak{O}_0^{(\alpha)}$. We can prove the latter assertions by considering the biholomorphic mapping $T\{z_1 = t^{p_2}, z_2 = t^{p_1}\}$ of the space $\{m, \mathfrak{O}^{(\alpha)}(m)\}$ on the disk $|t| < 1$. We write $\phi = t \circ T^{-1}$; this function is α-holomorphic on m, and its germ belongs to the ring $\mathfrak{O}_0^{(\alpha)}$; its germ cannot belong to the ring $\mathfrak{O}_0^{(\beta)}$, since the function $\sqrt[p_2]{z_1}$ has no continuous branches in the disk $|z_1| < 1$.

Let us define a set of β-structures in an arbitrarily chosen Hausdorff space R. If $\mathfrak{O}^{(\beta_1)}(R)$ and $\mathfrak{O}^{(\beta_2)}(R)$ are two structural bundles on the space R, every ring $\mathfrak{O}_r^{(\beta_2)}$ is a subring of the ring $\mathfrak{O}_r^{(\beta_1)}$ (for all $r \in R$) and if at some point $r_0 \in R$ we have the inequality $\mathfrak{O}_r^{(\beta_1)} \neq \mathfrak{O}_r^{(\beta_2)}$, the structural bundle $\mathfrak{O}^{(\beta_2)}(R)$ is called a *refinement* of the structural bundle $\mathfrak{O}^{(\beta_1)}(R)$.

In this case the identity mapping $i: (R, \mathfrak{O}^{(\beta_2)}(R)) \to (R, \mathfrak{O}^{(\beta_1)}(R))$ is a holomorphic mapping; the correspondence $i^*: \mathfrak{O}_r^{(\beta_1)} \to \mathfrak{O}_r^{(\beta_2)}$ (for every $r \in R$) is a homomorphic mapping of the ring $\mathfrak{O}_r^{(\beta_1)}$ into the ring $\mathfrak{O}_r^{(\beta_2)}$.

A structural bundle $\mathfrak{O}^{(\beta)}(R)$ not admitting a refinement is said to be *maximal*. In the same sense, we may speak of *maximal* β-structures. It is easy to see that a complex β-structure on a complex manifold is uniquely defined and is always maximal. This follows from the fact that homeomorphic and holomorphic mappings of a complex manifold on another complex manifold are always biholomorphic. We no longer distinguish between the complex manifold M and

the space $(M, \mathfrak{O}^{(\beta)}(M))$.

3. Properties of complex β-spaces. Primary importance in the characterization of a complex β-space attaches to the properties of the rings $\mathfrak{O}_r^{(\beta)}$ of germs of holomorphic functions at the points $r \in (R, \mathfrak{O}^{(\beta)}(R))$, which make up its structural bundle. We have the

THEOREM 16.1. *The ring $\mathfrak{O}_r^{(\beta)}$ is a Noether ring with a unique maximal ideal.*

PROOF. The ring $\mathfrak{O}_r^{(\beta)}$ is a homomorphic image, i.e., a factor-ring, of the ring $\mathfrak{O}_z$ of convergent power series centered at some point z of the space C^n. The latter ring is a Noether ring, in view of Theorem 4.8. Therefore the first part of the theorem follows from the fact that a factor-ring of a Noether ring is itself a Noether ring.

The unique maximal ideal of the ring $\mathfrak{O}_z$ is the ensemble of holomorphic functional germs vanishing at the point z. In fact, if the ideal I_z contains a function f not vanishing at z, it contains all functions $g \in \mathfrak{O}_z$, since in this case $g = f \cdot g/f \in I_z$, i.e., $I_z \equiv \mathfrak{O}_z$.

The second assertion of the theorem follows from the fact that a factor-ring has a unique maximal ideal if the original ring has one.

We note that for an arbitrary complex β-space the ring $\mathfrak{O}_r^{(\beta)}$ is not in general a ring of integrity.

We now note the following assertion, containing the classical principle of the maximum modulus for holomorphic functions in the space C^n (Theorem 3.11).

THEOREM 16.2. *Let the function $w = f(r)$, $w \in C_w^1$, $r \in (R, \mathfrak{O}^{(\beta)}(R))$ be holomorphic on some set $V \subset (R, \mathfrak{O}^{(\beta)}(R))$ and different from a constant in some neighborhood of the point $r_0 \in V$. Then this function maps every set $A \subset V$ for which r_0 is an inner point to the set $f(A) \subset C_w^1$, having $f(r_0)$ as an inner point.*

In formulating a number of properties of β-spaces we make essential use of the notion of β-chart.

If $U \subset (R, \mathfrak{O}^{(\beta)}(R))$, $B \subset C^n$ are open sets in their respective spaces, and $\psi\colon U \longrightarrow \psi(U) \subset B$ is a biholomorphic mapping of the set U on the analytic subset $\psi(U)$ of the set B, then the pair (U, ψ) is called a β-chart on the space $(R, \mathfrak{O}^{(\beta)}(R))$.

We assert the following:

1) Let R be a Hausdorff space, and $U_i \subset R$, $B_i \subset C^{n_i}$ be open sets in their

respective spaces; let $\psi_i: U_i \to \psi_i(U_i) \subset B_i$ be homeomorphic mappings of the sets U_i to the analytic subsets $\psi_i(U_i)$ of the sets B_i. Here $i \in I$, where I is some set of indices, and $\bigcup_{i \in I} U_i = R$. Then if the mapping $\psi_i \circ \psi_j^{-1}$: $\psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$, $i, j \in I$, is biholomorphic for all non-empty intersections $U_i \cap U_j$, we can define on the space R, and moreover define uniquely, a β-structural bundle $\mathfrak{D}^{(\beta)}(R)$ such that all pairs (U_i, ψ_i) are β-charts for the space $(R, \mathfrak{D}^{(\beta)}(R))$.

2) If $(R_1, \mathfrak{D}^{(\beta 1)}(R_1))$ and $(R_2, \mathfrak{D}^{(\beta 2)}(R_2))$ are two complex β-spaces, we can define on the space $R_1 \times R_2$ a unique β-structure having the following properties: if (U_1, ψ_1) and (U_2, ψ_2) are β-charts for the corresponding original spaces, then the pair $(U_1 \times U_2, \psi_1 \times \psi_2)$ is a β-chart for the β-space defined by these structures.

The dimension of the β-space $(R, \mathfrak{D}^{(\beta)}(R))$ at a point r is the complex dimension of the space R at the point r.

If at all points r of the analytic set $m \subset (R, \mathfrak{D}^{(\beta)}(R))$ the dimension satisfies $d_r(m) < d_r(R)$, the set m is nowhere dense in the space R; even in this case, however, it may split the space R.

If some point $r \in (R, \mathfrak{D}^{(\beta)}(R))$ has a pure-dimensional neighborhood $U_r \subset (R, \mathfrak{D}^{(\beta)}(R))$, the space is said to be *locally pure-dimensional at the point* r. In such a case there exists a β-chart (U^d, ψ), $r \in U^d$ on the space $(R, \mathfrak{D}^{(\beta)}(R))$, such that $\psi(U^d)$ is an analytic set on the polycylinder $Z\{|z_k| < 1,\ k = 1, \cdots, n\} \subset C^n$ and projects surjectively on the polycylinder $Z^d\{|z_k| < 1,\ k = 1, \cdots, d\}$.

We conclude this subsection with one more theorem, relating to holomorphic mappings of β-spaces.

THEOREM 16.3. *If* $T: (R_1, \mathfrak{D}^{(\beta 1)}(R_1)) \to (R_2, \mathfrak{D}^{(\beta 2)}(R_2))$ *is a proper holomorphic mapping of one complex* β*-space on another, the image* $T(R_1, \mathfrak{D}^{(\beta 1)}(R_1))$ *of the mapped space is an analytic set in the space* $(R_2, \mathfrak{D}^{(\beta 2)}(R_2))$.

4. Ordinary and exceptional points of complex β-spaces. Normalization of β-spaces.

DEFINITION. The point $r \in (R, \mathfrak{D}^{(\beta)}(R))$ is called an *ordinary* point of the space $(R, \mathfrak{D}^{(\beta)}(R))$ if there exists an open set $U_r \subset (R, \mathfrak{D}^{(\beta)}(R))$, $r \in U_r$, such that the space $(U_r, \mathfrak{D}^{(\beta)}(U_r))$ is a complex manifold.

(Thus, in this case the ring $\mathfrak{O}_r^{(\beta)}$ is isomorphic to the ring of convergent power series in the space C^{n_r} with center at the origin of coordinates.)

If such an open set U_r does not exist, the point r is called an exceptional point of the space $(R, \mathfrak{O}^{(\beta)}(R))$.

In subsection 5, §4, Chapter I we introduced the analogous notions of ordinary and exceptional points. We have the

THEOREM 16.4. *Let m be an analytic set in the open set $B \subset C^n$. The point $\zeta \in (m, \mathfrak{O}^{(\beta)}(m))$ is either an ordinary point of the space $(m, \mathfrak{O}(m))$ and of the analytic set m, or it is an exceptional point of both.*

PROOF. Let $\zeta \in m$ be an ordinary point of the set m and suppose that $d_\zeta(m) = \mu < n$. Then, by Theorem 2.3_2, with a suitable choice of the neighborhood $U_\zeta \subset C^n$ and a change (if necessary) in the order of numbering of the variables, the set $m \cap U_\zeta$ can be made to satisfy the equations

$$z_k - f_k(z_1, \cdots, z_\mu) = 0, \quad k = \mu + 1, \cdots, n. \tag{3.10}$$

Here the f_k are holomorphic functions on the open set $V_\zeta = \gamma(U_\zeta)$, where γ is a projection $(z_1, \cdots, z_n) \to (z_1, \cdots, z_\mu)$. It is evident that the mapping $\gamma|_{m \cap U_\zeta}$ is biholomorphic, and this proves our assertion for the space $(m, \mathfrak{O}^{(\beta)}(m))$.

We now assume that the intersection $m \cap U_\zeta$ is a complex manifold. Then there exists a biholomorphic mapping $\phi: W_\zeta \to \Omega$, where W_ζ is some neighborhood of the point ζ in the space $(m, \mathfrak{O}^{(\beta)}(m))$, and Ω is an open set in the space C^μ of the variables $\omega_1, \cdots, \omega_\mu$ $(\mu < n)$. The functions $\omega_k \circ \phi$ $(k = 1, \cdots, \mu)$ are holomorphic at the point $\zeta \subset (m, \mathfrak{O}^{(\beta)}(m))$; this means that they are traces of functions $h_k(z)$ holomorphic at the point $\zeta \in C^n$ with respect to the containing space C^n. We write the inverse image $\phi^{-1}: \Omega \to W_\zeta$ with the aid of the functions $z_k = g_k(\omega_1, \cdots, \omega_\mu)$, $k = 1, \cdots, n$, holomorphic in the open set Ω. Then the equations $\omega_k = (\omega_k \circ \phi) \circ \phi^{-1}$, $k = 1, \cdots, \mu$ can be written in the neighborhood of the point $\omega = \phi(\zeta) \in C^\mu$ in the form of the relations

$$\omega_k = h_k(g_1(\omega), \cdots, g_n(\omega)), \quad k = 1, \cdots, \mu. \tag{3.11}$$

We therefore find that

$$\sum_{s=1}^{n} \frac{\partial h_k}{\partial z_s} \frac{\partial g_s}{\partial \omega_l} = \begin{cases} 0 \ (k \neq l), \\ 1 \ (k = l). \end{cases} \tag{3.12}$$

Accordingly, the Jacobian matrix of the mapping ϕ^{-1} has the rank μ in the neighborhood of the point $\omega = \phi(\zeta)$. We may suppose (changing the numbering of the variables if necessary) that in fact $(\partial(z_1, \cdots, z_\mu)/\partial(\omega_1, \cdots, \omega_\mu))_\omega \neq 0$. It follows that the system of functions $z_k = g_k(\omega_1, \cdots, \omega_\mu)$ admits a holomorphic mapping in the neighborhood of the point $\omega = \phi(\zeta)$, and therefore that the set m can be represented in some neighborhood of the point ζ by equations of the form (3.10). The assertion of our theorem has been proved for the set m.

The following theorems are also valid.

THEOREM 16.5. *If $T: (R, \mathfrak{O}^{(\beta)}(R)) \to (S, \mathfrak{O}^{(\beta_1)}(S))$ is a biholomorphic mapping of one complex β-space on another, the points $r \in (R, \mathfrak{O}^{(\beta)}(R))$ and $Tr \in (S, \mathfrak{O}^{(\beta_1)}(S))$ are both ordinary points or both exceptional points in their respective spaces.*

THEOREM 16.6. *The ensemble N of exceptional points of the complex β-space $(R, \mathfrak{O}^{(\beta)}(R))$ is always an analytic set, and at all points $r \in N$ the dimension satisfies $d_r(N) < d_r(R, \mathfrak{O}^{(\beta)}(R))$. The set $R \setminus N$ is a complex manifold.*

We consider the point r of the space $(R, \mathfrak{O}^{(\beta)}(R))$ and a β-map (U, ψ), $r \in U$ on this space. At the point $\psi(r) \in C^n$ there belong to the analytic set $\psi(U) \subset C^n$ a number $k(r)$ of simple germs of analytic sets (cf. Theorem 4.10). To these simple germs at the point r of the space $(R, \mathfrak{O}^{(\beta)}(R))$ there correspond $k(r)$ simple germs p_r of the space $(R, \mathfrak{O}^{(\beta)}(R))$. The quantity $k(r)$ is called the *number of simple germs* of the space $(R, \mathfrak{O}^{(\beta)}(R))$ at the point r.

DEFINITION (normalization of a complex β-space). Let $(R, \mathfrak{O}^{(\beta)}(R))$ be a complex β-space and N the set of its exceptional points. The pair (R^*, ρ) is called a normalization of the space $(R, \mathfrak{O}^{(\beta)}(R))$ if:

1) R^* is a locally compact and locally connected space, and the mapping $\rho: R^* \to R$ is continuous, nowhere exceptional, proper, and surjective.

2) The set $\rho^{-1}(N)$ nowhere splits the space R^*, and the restriction $\rho: R^* \setminus \rho^{-1}(N) \to R \setminus N$ is a homeomorphic mapping.

THEOREM 16.7. *A complex β-space is always normalizable.*

PROOF. We denote by R^* the set of all simple germs p_r for the points $r \in (R, \mathfrak{O}^{(\beta)}(R))$, where $(R, \mathfrak{O}^{(\beta)}(R))$ is some β-space. The mapping $\rho: p_r \to r$ will be called the natural projection of the set R^* on the space R. Let U be an analytic set in the open set $B \subset (R, \mathfrak{O}^{(\beta)}(R))$. We denote by $p_{r_0}^U$

those simple germs of analytic sets of the space $(R, \mathfrak{O}^{(\beta)}(R))$ at the point r_0 that belong to the set U. We agree to call an open subset of the space R^* a set of the form $U^* = \{p_r^U, r \in U\}$. As a result, a topology is introduced in the set R^* and the set becomes a topological space. It is clear that in this topology the projection $\rho: R^* \to R$ is a continuous nowhere exceptional mapping.

Considering the β-maps of the space $(R, \mathfrak{O}^{(\beta)}(R))$, we establish the fact that the space R^* can be covered by open sets U_j^* (where $j \in J$ and J is a set of indices), each of which is homeomorphic to the normalization space m_j^* for some analytic set m_j in the open set $B_j \subset C^{n_j}$. It follows, in view of Theorem 14.3, that the space R^* is locally compact and locally connected, that the set $\rho^{-1}(N)$ nowhere splits R^*, and the restriction $\rho/R^* \setminus \rho^{-1}(N)$ is a homeomorphic mapping.

The pair (R^*, ρ) is a normalization of the space $(R, \mathfrak{O}^{(\beta)}(R))$.

In what follows, when speaking of the normalization of a complex β-space, we shall mean the pair (R^*, ρ) constructed by the process given in the proof of Theorem 16.7.

Further properties of the normalization of complex β-spaces are given in the following theorem.

THEOREM 16.8. 1) *If (R^*, ρ) and $('R^*, '\rho)$ are two normalizations of the complex β-space $(R, \mathfrak{O}^{(\beta)}(R))$, there exists a homeomorphic mapping $\tau: 'R^* \to R^*$, such that $'\rho = \rho \circ \tau$.*

2) *If (R^*, ρ) is a normalization of the space $(R, \mathfrak{O}^{(\beta)}(R))$, the set $\rho^{-1}(r)$ for every $r \in (R, \mathfrak{O}^{(\beta)}(R))$ consists of $k(r)$ points. If m is an analytic set in the space $(R, \mathfrak{O}^{(\beta)}(R))$ and for all points $r \in m$ the dimension satisfies $d_r(m) < d_r(R)$, then the set $\rho^{-1}(m)$ nowhere splits the space R^*.*

3) *If R_k^* $(k = 1, \cdots, K)$ is a connected component of the space R^*, then $\rho(R_k^*) = R_k$ is an analytic pure-dimensional set in the space $(R, \mathfrak{O}^{(\beta)}(R))$. Every set $R_k \setminus N$ is a manifold.*

The β-spaces $(R_k, \mathfrak{O}^{(\beta)}(R_k))$ are called *irreducible* components of the space $(R, \mathfrak{O}^{(\beta)}(R))$. The space $(R, \mathfrak{O}^{(\beta)}(R))$ is called *irreducible* (in the large) if it is its only irreducible component.

5. **Complex β_i-spaces.**

DEFINITION. A complex β-space $(R, \mathfrak{O}^{(\beta)}(R))$ is called *irreducible at a point* r if the corresponding ring $\mathfrak{O}_r^{(\beta)} \in \mathfrak{O}^{(\beta)}(R)$ is an integral domain. In this

case r is a *point of irreducibility* or, for short, an *i-point* of the space. A complex β-space consisting altogether of i-points is called a *β_i-space* and its structural bundle $\mathfrak{O}_i^{(\beta)}(R)$ is called a *β_i-structural bundle.*

THEOREM 16.9. *The space $(R, \mathfrak{O}^{(\beta)}(R))$ is irreducible at the point r if and only if $k(r) = 1$. The space is a β_i-space if and only if the mapping $\rho: R^* \longrightarrow R$ is homeomorphic.*

PROOF. We consider the β-chart (U, ψ) where $r \in U$, the analytic set $\psi(U) \subset C^n$ and the germ of the analytic set $\psi(r) \subset C^n$ that represents it. Let $I_{\psi(r)}$ be the ideal corresponding to this germ in the ring $\mathfrak{O}_{\psi(r)}$. Then the ring $\mathfrak{O}_r^{(\beta)} \in \mathfrak{O}^{(\beta)}(R)$ is isomorphic to the ring $\mathfrak{O}_{\psi(r)} \mid I_{\psi(r)}$. The latter has no zero divisors if and only if the set $\psi(U)$ is irreducible at the point $\psi(r)$, that is, if $k(r) = 1$. The second assertion of the theorem is therefore valid.

We have noted above that the analytic set $(m, \mathfrak{O}(m))$ is always a complex β-space in some open set $D \subset C^n$. *If and only if this set is locally irreducible, the space $(m, \mathfrak{O}(m))$ is a β_i-space.*

It follows from the general properties of irreducible analytic sets and from Theorems 16.4, 16.5, and 16.9 that an analytic set of minimum dimension nowhere splits a β_i-space. In order that the space $(R, \mathfrak{O}^{(\beta)}(R))$ be a β_i-space it is necessary and sufficient that it not be split by the set of its exceptional points.

From this follows

THEOREM 16.10. *A β-structure, defined on an arbitrary (topological) space is always a β_i-structure.*

PROOF. We may limit ourselves to the consideration of a connected space R. If $2m$ is its (topological) dimension, the dimension of the set N of exceptional points of the β-space $(R, \mathfrak{O}^{(\beta)}(R))$ is not greater than $2m - 2$. From this it follows [1] that the set N nowhere splits the space $(R, \mathfrak{O}^{(\beta)}(R))$, and therefore it is a β_i-space.

THEOREM 16.11. *If the bundle of rings $\mathfrak{O}(R)$ defines a β_i-structure on the Hausdorff space R, every β-structure $\mathfrak{O}'(R)$ that refines $\mathfrak{O}(R)$ is also a β_i-structure.*

1) Cf., for example, W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Mathematical Series, Vol. 4, Princeton Univ. Press, Princeton, N. J., 1941; Russian transl., IL, Moscow, 1948, p. 74.

PROOF. We consider the set N' of exceptional points of the space $(R, \mathfrak{D}'(R))$. The identity mapping $i: (R, \mathfrak{D}') \to (R, \mathfrak{D})$ is clearly both homeomorphic and holomorphic. Therefore, the set $N' = i(N')$ is an analytic set of the β_i-space $(R, \mathfrak{D}(R))$ of dimension not lower than that of the whole space (cf. Theorem 16.3). It follows that the set N' nowhere splits the space $(R, \mathfrak{D}(R))$, and so nowhere splits the space $(R, \mathfrak{D}'(R))$. This completes the proof of Theorem 16.11.

§17. NORMAL SPACES OF H. CARTAN

1. Fundamental concept.

DEFINITION (normal space of H. Cartan). The complex β-space $(R, \mathfrak{D}^{(\beta)}(R))$ is called normal at one of its points r if the corresponding ring $\mathfrak{D}_r^{(\beta)} \in \mathfrak{D}^{(\beta)}(R)$ is integrally closed in its quotient field. In this case the point r is called a *normal* point of the space, or for short an *n-point.* A complex β-space consisting entirely of n-points is called a β_n-space or also a normal space of H. Cartan; its structural bundle is called a β_n-structural bundle.

If R is a complex manifold its structural β-bundle $\mathfrak{D}(R)$ has the property under discussion; therefore a complex manifold is always a β_n-space.

DEFINITION (normal analytic set). An analytic set m of the complex manifold R is called *normal* at the point $r \in m$ if r is a normal point of the complex space $(m, \mathfrak{D}(m))$. An analytic set m is called normal in the manifold R (or normally imbedded in the manifold R) if it is normal at all points of the manifold. Here $\mathfrak{D}(m)$ is the structural bundle induced on the set m by the structural bundle of the manifold R.

THEOREM 17.1. *Every n-point of a complex β-manifold is an i-point of it, and a complex β_n-space is a complex β_i-space.*

PROOF. We assume, contrary to our hypothesis, that there exists an n-point $r \in (R, \mathfrak{D}^{(\beta)}(R))$ and that $k(r) > 1$. Then there exists on the space $(R, \mathfrak{D}^{(\beta)}(R))$ a β-chart (U, ψ), $r \in U$, for which the analytic subset $\psi(U)$ of the open set $D \subset C_z^n$ decomposes into two analytic sets m_1 and m_2 distinct from $\psi(U)$, with $r \in m_1 \cap m_2$.

We denote by $m_1^{(k_1)}$, $m_2^{(k_2)}$ (where $k_1, k_2 = 1, 2, \cdots$) two irreducible analytic sets containing the point $\psi(r)$ and contained in the sets m_1 and m_2. Under these conditions it is always possible to find functions $f_1(z)$, $f_2(z)$,

holomorphic on the open set D and satisfying the relationships

$$\left.\begin{array}{l} f_1 \begin{cases} \equiv 0 \text{ for } z \in m_1 \\ \not\equiv 0 \text{ for } z \in m_2^{(k_2)} \text{ (for every } k_2) \end{cases} \\ f_2 \begin{cases} \not\equiv 0 \text{ for } z \in m_1^{(k_1)} \text{ (for every } k_1) \\ \equiv 0 \text{ for } z \in m_2. \end{cases} \end{array}\right\} \qquad (3.1$$

Let $\tilde{f}_1$ and $\tilde{f}_2$ be the traces of these functions on that analytic set $\psi(U)$. Then $\tilde{f}_1 \cdot \tilde{f}_2 \equiv 0$, while $\tilde{f}_1 + \tilde{f}_2$ is not a divisor of zero in the ring $\mathfrak{O}_{\psi(r)} \in \mathfrak{O}(\psi(U))$. (This function is distinct from the function that is identically equal to zero on all irreducible components of the analytic set $\psi(U)$ which contain the point $\psi(r)$.) It follows that the function $v = \tilde{f}_1 / (\tilde{f}_1 + \tilde{f}_2)$ belongs to the quotient ring of the ring $\mathfrak{O}_{\psi(r)} \in \mathfrak{O}(\psi(U))$. The function v is an algebraic integer of this ring since it satisfies the equation $w^2 - w = 0$. It follows that $v \in \mathfrak{O}_{\psi(r)}$. But this is impossible since

$$v \equiv \begin{cases} 0 \text{ on } m_1, \\ 1 \text{ on } m_2 \end{cases} \qquad (3.14)$$

and therefore the function v has a discontinuity at the point $\psi(r)$.

COROLLARY. *A normal analytic subset of a complex manifold is always locally irreducible.*

2. **The class F of complex spaces.**

THEOREM 17.2. *The class of complex β_n-spaces coincides with the class F of complex β-spaces.*

PROOF. 1) Let $(R, \mathfrak{O}^{(\beta)}(R))$ be a β-space of class F, let r be a point of this space, and let $h_r = f_r / g_r$ (where $f_r, g_r \in \mathfrak{O}_r$) be an algebraic integer in the quotient ring of the ring $\mathfrak{O}_r$. Then in a sufficiently small neighborhood U_r of the point r:

a) there exist holomorphic functions f and g, representing germs f_r and g_r; suppose that $N \subset U_r$ is the set, nowhere dense in U_r, consisting of the zeros of the function g;

b) at the points $U_r \setminus N$ the ratio f/g satisfies the equation

$$w^m + \sum_{\mu=1}^{m} a_\mu(r)\, w^{m-\mu} = 0 \qquad (3.15)$$

with coefficients a_μ holomorphic in U_r.

It follows that the function f/g is bounded in the neighborhood of every point of the set N and, since the space $(R, \mathfrak{D}^{(\beta)}(R))$ belongs to the class F, can be extended to a function h^* holomorphic in the whole neighborhood U_r. Then throughout this neighborhood $f = h^*g$, and therefore also $f_r = h_r^* \cdot g_r$, where $h_r^* \in \mathfrak{D}_r$ is the germ of a holomorphic function representable in the neighborhood U_r by the function h^*. In view of the general properties of a quotient ring $h_r \equiv h_r^*$ and therefore $h_r \in \mathfrak{D}_r$. Accordingly the point r is a normal point of the space $(R, \mathfrak{D}^{(\beta)}(R))$. We have shown that the space consists of normal points and is thus a β_n-space.

2) We now assume that $(R, \mathfrak{D}^{(\beta)}(R))$ is a complex β_n-space and show that Theorem 6.7 on the extension of functions holds for it. Let r be an arbitrary point of this space, let U_r be a sufficiently small neighborhood of the point, let τ be a biholomorphic mapping of the neighborhood U_r on some analytic subset M of the region G of the space C^n, and finally, let $\mathfrak{D}(M)$ be the structural bundle of rings of holomorphic functions on M. By the preceding theorem, the set M is locally irreducible; since U_r is small, we may assume that it is irreducible in G and therefore that it has a pure dimension $d(M) = d < n$, and the region G itself is a product of the regions $G^d \subset C^d$, $G^{n-d} \subset C^{n-d}$, while the projection $\gamma: M \longrightarrow G^{n-d}$ is surjective. Then by Theorem 14.4 the triple $\mathfrak{M} = (M, \gamma, G^d)$ defines an analytic covering of the region G^d.

We now consider an analytic set $N \subset (M, \mathfrak{D}(M))$ with codimension not less than unity. Clearly, it is also analytic relative to the covering $\mathfrak{M}$. Let $f|_{M \setminus N}$ be a holomorphic function bounded in some neighborhood of every point of the set N. From Riemann's theorem on the extension of analytic functions, which holds for analytic coverings, it follows that the function $f|_{M \setminus N}$ can be extended to a function $\hat{f}|_M$, holomorphic relative to the covering $\mathfrak{M}$. We show that this function is holomorphic in M relative to the structural bundle $\mathfrak{D}(M)$. To this end it is sufficient (since we are dealing with a β_n-space) to establish that for every point $z_0 \in M$ the function $\hat{f}$ belongs to the germ $\hat{f}_{z_0}$, which in turn belongs to the quotient ring of the ring $\mathfrak{D}_{z_0} \in \mathfrak{D}(M)$ and is an algebraic integer for the ring $\mathfrak{D}_{z_0}$. The fact, however, that the germ $\hat{f}_{z_0}$ belongs under the given conditions to the quotient ring of the ring $\mathfrak{D}_{z_0}$ is a consequence of of Theorem 15.2. The fact that it is an algebraic integer for the ring $\mathfrak{D}_{z_0}$ follows from Theorem 15.1. In view of this theorem, the function $\hat{f}$ satisfies the equation

$$w^b + \sum_{\beta=1}^{b} a_\beta(z)\, w^{b-\beta} = 0 \tag{3.16}$$

with coefficients $a_\beta(z)$ holomorphic in G^d; each of these functions $a_\beta(z)$ defines a germ $(a_\beta)_{z_0}$ belonging to the ring $\mathfrak{O}_{z_0} \in \mathfrak{O}(M)$. The germ $\hat{f}_{z_0}$ satisfies equation (3.16) with these coefficients.

THEOREM 17.3. *A homeomorphic and holomorphic mapping* $r\colon (R_1, \mathfrak{O}^{(\beta_1)}(R_1)) \to (R_2, \mathfrak{O}^{(\beta_2)}(R_2))$, *where* $(R_1, \mathfrak{O}^{(\beta_1)}(R_1))$ *is a* β*-space and* $(R_2, \mathfrak{O}^{(\beta_2)}(R_2))$ *is a* β_n*-space, is always biholomorphic.*

PROOF. We are to prove that the mapping r^{-1} is holomorphic. Let N_1 and N_2 be the sets of exceptional points of the spaces $(R_1, \mathfrak{O}^{(\beta_1)}(R_1))$ and $(R_2, \mathfrak{O}^{(\beta_2)}(R_2))$. Then by Theorem 16.6 these sets are analytic, and we have $d_r(N_1) < d_r(R_1, \mathfrak{O}^{(\beta_1)}(R_1))$ and $d_r(N_2) < d_r(R_2, \mathfrak{O}^{(\beta_2)}(R_2))$ at all points r of these spaces. Then the mapping $r^{-1}\colon R_2 \setminus (N_2 \cup r(N_1)) \to R_1 \setminus (r^{-1}(N_2) \cup N_1)$ is holomorphic, since it is the inverse of a homeomorphic and holomorphic mapping of one complex manifold on another. By the preceding theorem the space $(R_2, \mathfrak{O}^{(\beta_2)}(R_2))$ belongs to the class F, and therefore the mapping r^{-1} is holomorphic on the whole space $(R_2, \mathfrak{O}^{(\beta)}(R_2))$.

In consequence of this theorem, a β_n-structure $\mathfrak{O}^{(\beta)}(R)$ on an arbitrary Hausdorff space R is always maximal.

We agree to say that the complex β-space $(R, \mathfrak{O}^{(\beta)}(R))$ belongs to the class $\tilde{F}$ if every function f which is continuous on an open set $U \subset (R, \mathfrak{O}^{(\beta)}(R))$ and holomorphic on the set $U \setminus N$, where N is an analytic subset of the set U with a codimension at least equal to unity, is holomorphic on the whole set U.

It is sometimes said that β-spaces of class $\tilde{F}$ are characterized by the fact that the weak form of Riemann's theorem on analytic extension of functions holds for them.

Clearly, every β-space of class F is a space of class $\tilde{F}$. On the other hand, a β-space of class $\tilde{F}$ does not necessarily belong to the class F. For example, consider the β-space defined on the analytic set $m = \{z_1 z_2 = 0\} \subset C^2$ by the induced structure $\mathfrak{O}(m)$.

We have the

THEOREM 17.4. *A complex* β_i*-space is a* β_n*-space if and only if it*

belongs to the class $\widetilde{F}$.

PROOF. It is sufficient to show that if a β_i-space $(R, \mathfrak{O}^{(\beta_i)}(R))$ belongs to $\widetilde{F}$ it also belongs to F.

Assume that we are given the open set $U \subset (R, \mathfrak{O}^{(\beta_i)}(R))$, that the set $N \subset U$ is nowhere dense in the set U, and that the function $f \mid U \setminus N$ is holomorphic on the set $U \setminus N$ and bounded in the neighborhood of every point of N. Then this function (see subsection 3 of the Introduction) can be continuously extended and also since the space $(R, \mathfrak{O}^{(\beta_i)}(R))$ belongs to the class $\widetilde{F}$, holomorphically extended to the whole set U. The theorem is proved.

3. Normalization of complex β-spaces. The basic role belongs to a theorem proved by K. Oka. [1)]

THEOREM 17.5. *Let* m *be an analytic subset of an open set in the space* C^p, $z \in m$ *a point in this subset, and* (m^*, μ) *a normalization of* m. *Then there exist:* 1) *a neighborhood* U_z *of the point* z; *and* 2) *a homeomorphic mapping* $\tau: [m^* \cap (\mu^{-1} U_z)] \longrightarrow \widetilde{m}$, *where* $\widetilde{m}$ *is a normal analytic subset of some open set in the space* C^q, *such that the mapping* $\mu \circ \tau^{-1}: \widetilde{m} \longrightarrow m$ *is holomorphic. Here* p *and* q *are arbitrary natural numbers.*

The following theorem is a consequence.

THEOREM 17.6. *If the pair* (R^*, ρ) *is a normalization of the complex* β*-space* $(R, \mathfrak{O}^{(\beta)}(R))$, *then on the Hausdorff space* R^* *it is possible to define a unique* β_n*-structure such that the mapping* $\rho: R^* \longrightarrow R$ *is holomorphic.*

REMARK. In Theorem 17.6 the normalization (R^*, ρ) of the space $(R, \mathfrak{O}^{(\beta)}(R))$ is constructed by the process given in the proof of Theorem 16.7.

PROOF. The existence of the desired β_n-structure is guaranteed by Theorem 17.5 (Oka's theorem); here we prove only its uniqueness.

Let $\mathfrak{O}(R^*)$ and $\mathfrak{O}'(R^*)$ be two β_n-structural bundles defined on the space R^*. We denote by N the set of exceptional points of the space R. Then the set $\rho^{-1}(N)$ is an analytic set relative to both structures. Since $R \setminus N$ is a manifold, the mapping $\rho: R^* \setminus \rho^{-1}(N) \longrightarrow R \setminus N$ is biholomorphic relative to both structures. It follows that the identity mapping $i = \rho^{-1} \circ \rho: (R^*, \mathfrak{O}(R^*)) \longrightarrow (R^*, \mathfrak{O}'(R^*))$ is biholomorphic outside the set $\rho^{-1}(N)$. Since the spaces

1) See Oka [6]. In this connection, see also a remark by Thimm [2] and a paper by Kuhlmann [1].

$(R^*, \mathfrak{O}(R^*))$ and $(R^*, \mathfrak{O}'(R^*))$ belong to the class F, the mapping i may be extended as a biholomorphic mapping to the whole space R^*. Clearly then the structures $\mathfrak{O}(R^*)$ and $\mathfrak{O}'(R^*)$ coincide.

THEOREM 17.7. *A β_i-structural bundle $\mathfrak{O}^{(\beta_i)}(R)$ defined on a Hausdorff space R, may always be refined, and moreover uniquely refined, to a normal β_n-structural bundle $\mathfrak{O}^{(\beta_n)}(R)$ on the same space.*

If the analytic set $m \subset C^n$ is locally irreducible, the projection $\mu: m^* \to m$ is a homeomorphism. We consider the normal structure $\mathfrak{O}^{(\beta_n)}(m^*)$ defined uniquely on the space m^* by virtue of Theorems 17.5 and 17.6. Beginning with this structure, we use the projection μ to define a normal structure $\mathfrak{O}^{(\beta_n)}(m)$ on the set m. This structure is often called the *natural complex structure* of the set m. We do not usually distinguish the normal complex space $(m, \mathfrak{O}^{(\beta_n)}(m))$ from the analytic set m and we denote both by the same symbol.

If m is a normal analytic set, its induced (by the containing space C^n) structural bundle $\mathfrak{O}(m)$ satisfies $\mathfrak{O}(m) = \mathfrak{O}^{(\beta_n)}(m)$. In the general case $\mathfrak{O}(m) \subset \mathfrak{O}^{(\beta_n)}(m)$. Osgood's theorem (Theorem 15.2) implies that the germs of β_n-holomorphic functions belonging to the ring $\mathfrak{O}_z^{(\beta_n)} \in \mathfrak{O}^{(\beta_n)}(m)$, where $z \in m$, are always traces on the set m of germs of meromorphic functions F_z on the space [1] C^n.

4. **The relationship between α-spaces and β-spaces.** We call the complex β-space $(R, \mathfrak{O}^{(\beta)}(R))$ a complex α-space if its β-structural bundle $\mathfrak{O}^{(\beta)}(R)$ is an α-structural bundle on the space R. The inverse relationship is similarly defined.

THEOREM 17.8. *A complex β_n-space is always a complex α-space.*

PROOF. Let $(R, \mathfrak{O}^{(\beta_n)}(R))$ be a complex β_n-space and $r \in (R, \mathfrak{O}^{(\beta_n)}(R))$. We are to show that there exists a neighborhood $U \subset (R, \mathfrak{O}^{(\beta_n)}(R))$ of the point r such that the restriction $\mathfrak{O}^{(\beta_n)}(R)|_U = \mathfrak{O}^{(\beta_n)}(U)$ is an α-structural bundle on the space U.

The neighborhood U may be assumed to be pure-dimensional, and we write $U = U^d$. As we proved in subsection 3 of §16, there exists under these circumstances a β_n-chart (U^d, ψ) on the neighborhood U^d for which the

1) See, along these lines, Hitotumatu [1], Thimm [1].

projection $\gamma: M \to Z^d$ of the locally irreducible analytic set $M = \psi(U) \subset Z\{|z_k| < 1,\ k = 1, \cdots, \nu\} \subset C_z^\nu$ on the polycylinder $Z^d\{|z_k| < 1,\ k = 1, \cdots, d\}$ is surjective. Then the triple $\mathfrak{M} = \{M, \gamma, Z^d\}$, by virtue of Theorem 14.4, turns out to be an analytic covering of the polycylinder Z^d, and the triple $(U^d, \psi, \mathfrak{M})$ is an α-map on the space U^d. With the aid of this α-map we define on the space U an α-structural bundle $\mathfrak{O}^{(\alpha)}(U)$. Theorem 17.8 will have been proved if we show that $\mathfrak{O}^{(\beta_n)}(U) = \mathfrak{O}^{(\alpha)}(U)$.

Let $A \subset Z^d$ (but $A \neq Z^d$) be an analytic set containing all γ-projections of branch points of the analytic covering $\mathfrak{M}$. Then the set $\gamma^{-1}(A)$ contains all exceptional points of the analytic set M, and the set $D = \psi^{-1}(\gamma^{-1}(A)) \subset U$ is analytic relative to both structural bundles $\mathfrak{O}^{(\beta_n)}(U)$ and $\mathfrak{O}^{(\alpha)}(U)$ and has codimension not less than unity. We write $\mathfrak{O}^{(\beta_n)}(R)\,|_{U \setminus D} = \mathfrak{O}^{(\beta_n)}(U \setminus D)$, $\mathfrak{O}^{(\alpha)}(U)\,|_{U\setminus D} = \mathfrak{O}^{(\alpha)}(U \setminus D)$. Clearly, $\mathfrak{O}^{(\beta_n)}(U \setminus D) \subset \mathfrak{O}^{(\alpha)}(U \setminus D)$. We consider the arbitrary open set $W \subset M \setminus \gamma^{-1}(A)$. It is evident that this set consists of ordinary points of the set M. Every function holomorphic on this set relative to the analytic covering $\mathfrak{M}$ may be considered locally as the trace of a function holomorphic in the corresponding part of the space C_z^ν. It follows that it is holomorphic relative to the β_n-space $(W, \mathfrak{O}(W))$. Therefore

$$\mathfrak{O}^{(\beta_n)}(U \setminus D) \supset \mathfrak{O}^{(\alpha)}(U \setminus D)$$

and accordingly

$$\mathfrak{O}^{(\beta_n)}(U \setminus D) = \mathfrak{O}^{(\alpha)}(U \setminus D).$$

Clearly the identity mapping

$$i: (U, \mathfrak{O}^{(\beta_n)}(U)) \to (U, \mathfrak{O}^{(\alpha)}(U))$$

is biholomorphic outside the set D. The set D is analytic in both complex spaces; both these spaces belong to the class F. It follows (see subsection 1 of this section) that the mapping i is biholomorphic on the whole space U. Then the equation $\mathfrak{O}^{(\beta_n)}(U \setminus D) = \mathfrak{O}^{(\alpha)}(U \setminus D)$ implies that $\mathfrak{O}^{(\beta_n)}(U) = \mathfrak{O}^{(\alpha)}(U)$. The proof is complete.

THEOREM 17.9. *A ring structure $\mathfrak{O}(R)$ on a Hausdorff space R is a β_n-structure if and only if it is an α_c-structure.*

PROOF. 1) We show that an arbitrarily chosen β_n-structural bundle $\mathfrak{O}^{(\beta_n)}(R)$, defined on the Hausdorff space R, is an α_c-structural bundle. That

it is an α-structural bundle follows from the preceding theorem. Suppose $r_0 \in R$, $U \subset R$ a neighborhood of r_0, and (U, ψ, γ) an α-chart on the space R. We choose U small enough so that there exists a biholomorphic mapping $\phi: U \to V$, where V is a normal analytic set in the region $B \subset C^n$.

We assume that the mapping ϕ is defined by the functions $z_\nu = f_\nu^*(r)$, $r \in U$, $\nu = 1, \cdots, n$ and that the analytic covering γ is defined by the triple $\gamma = (Y, \eta, G)$.

We write $f_\nu(y) = f_\nu^* \circ \psi^{-1}$, where $y \in Y$; since the mapping $\psi: U \to Y$ is biholomorphic, the functions $f_\nu(y)$ are holomorphic. Since ϕ is a biunique mapping, there exists for any two arbitrary points $y_1, y_2 \in Y$ (where $y_1 \neq y_2$) a number ν_0 such that $f_{\nu_0}(y_1) \neq f_{\nu_0}(y_2)$. It follows that for every point $\zeta \in G$ we may choose a function $f(y)$ holomorphic in Y and taking on different values at the two arbitrarily chosen points $y_1, y_2 \in \eta^{-1}(\zeta)$. This function $f(y)$ clearly satisfies the equation $[f: \mathfrak{D}(G)] = b(y)$. We have therefore shown that the analytic covering γ is algebroidal (see the definition given in subsection 1, §15), and the β_n-structure under consideration is an α_c-structure.

2) We now consider some α_c-structural bundle $\mathfrak{D}^{(\alpha_c)}(R)$ on the space R. Suppose that the point $r_0 \in R$, that $U \subset R$ is a neighborhood of this point, that (U, ψ, γ) is an α-chart on the space R, that $\gamma = (Y, \eta, G)$ is an algebroidal analytic covering, and that $\mathfrak{D}^{(\alpha)}(Y)$ is its α-structural bundle. The region $G \subset C_z^n$ is to be chosen so small that there exists in the space Y a holomorphic function $w = f(y)$ for which $[f: \mathfrak{D}(G)] = b(y)$; this function satisfies the equation

$$\omega(w, z) = w^b + a_1(z) w^{b-1} + \cdots + a_b(z) = 0 \tag{3.17}$$

with coefficients $a_k(z)$ $(k = 1, \cdots, b)$ holomorphic in the region G (see subsection 1, §15).

We consider the analytic set $M = \{\omega(w, z) = 0\} \subset G \times C_w^1$ and the mapping $\phi: Y \to M$, defined by the equation $\phi = \eta \times \{w = f(y)\}$. If $D \subset G$ is the discriminant set of the pseudopolynomial $\omega(w, z)$, the restriction $\phi|_Y: Y_0 \to M_0$, where $Y_0 = Y \setminus \eta^{-1}(D)$, $M_0 = M \setminus D \times C_w^1$, is a biholomorphic mapping. Since: a) the codimension of the set $D \times C_w^1 \subset M$ is not less than unity; b) ϕ is a proper and nowhere exceptional mapping; and c) an analytic subset that has in an open set of the space Y a codimension not less than unity nowhere splits that open set, it follows that the pair (Y, ϕ) is a normalization of the set M. Then in view of Theorem 16.6

we may define on the space Y a β_n-structural bundle $\mathfrak{D}^{(\beta_n)}(Y)$ such that the mapping $\phi: (Y, \mathfrak{D}^{(\beta_n)}(Y)) \to (M, \mathfrak{D}(M))$ is holomorphic relative to it, and the restriction of the identity mapping $i|_{Y_0}: (Y_0, \mathfrak{D}^{(\beta_n)}(Y)) \to (Y_0, \mathfrak{D}^{(\beta_n)}(Y_0))$ is biholomorphic. Then since α-spaces and β_n-spaces belong to the class F, it follows that the mapping i is biholomorphic on the whole space Y. Accordingly, the given α_c-structural bundle defines a β_n-structure on the space U.

Theorem 17.9 and Theorem 15.4 (the Grauert-Remmert theorem) imply

THEOREM 17.10. *Every α-space is a β_n-space.*

Theorems 17.8 and 17.10 show that the class of α-spaces (Behnke-Stein spaces) coincides with the class of β_n-spaces (normal spaces of H. Cartan).

From now on we will as a rule consider only those normal β_n-spaces = complex α-spaces that consist of an at most countable set of components. We shall not, in general, indicate explicitly that our complex spaces R are of this nature. On the other hand, whenever we wish to consider spaces of a more general kind, we will always say so explicitly. [1)]

§18. HOLOMORPHICALLY COMPLETE SPACES AND MANIFOLDS

1. Complex manifolds without a countable basis of open sets. In view of Rado's classical theorem, a complex analytic manifold of one complex dimension always has a countable basis of open sets, that is, the second axiom of countability always holds for such a manifold. On the other hand, an arbitrary topological two-dimensional real analytic manifold does not necessarily have a countable basis of open sets. A counterexample, going back to Prufer, was also given by Rado.

Using the idea underlying the Prufer-Rado example, E. Calabi and M. Rosenlicht [1] have constructed a complex analytic manifold of complex dimension $n > 1$ having no countable basis of open sets.

We consider first the Prufer-Rado manifold. We set up a correspondence between the numbers $t \in R$, where R is the set of all natural numbers, and Euclidean planes E_t; we let x_t, y_t be the Cartesian coordinates of a point in the plane E_t. We shall say that the points $(x_s, y_s) \in E_s$, $(x_t, y_t) \in E_t$, are equivalent if

1) Further results relating to the general theory of complex spaces may be found in: Grauert-Remmert [2], H. Cartan [6, 7].

$$y_s = y_t,$$
$$x_s y_s + s = x_t y_t + t \text{ for } y_s = y_t > 0. \tag{3.18}$$

We denote by S_t the set of classes of equivalent points of this kind; every such class is called a point of the set S_t. Let $V_t \subset S_t$ be the set of points of S_t corresponding to the points of the plane E_t. We introduce into each V_t the topology of the plane E_t; it is easy to see that the topologies so induced in the spaces V_t and V_s are compatible in the intersection $V_t \cap V_s$. Thus, a Hausdorff topological structure is defined on the whole set $S_t = \bigcup_{t \in R} V_t$. The set S_t is, in this topology, a real analytic manifold (a topological two-dimensional real analytic surface). The spaces V_t play the role of maps on this manifold; the local coordinates in these maps are the x_t, y_t — the coordinates of the points of the corresponding space E_t.

The surface S_t so constructed was considered in the Prufer-Rado example.

Aside from the surfaces S_t, we consider another surface S. The latter is defined, starting with the set of planes E_t, $t \in R$, in the same way as the surfaces S_t, but using the equivalence conditions

$$\begin{aligned} y_s &= y_t; \\ x_s y_s + s &= x_t y_t + t \quad \text{for } s \neq t; \\ x_s &= x_t \quad\quad\quad\quad \text{for } s = t. \end{aligned} \tag{3.19}$$

It is easy to see that the real analytic surfaces S_t and S have no countable basis of open sets. This follows immediately from the fact that in the surfaces S_t and S there is an uncountable discrete set of points corresponding to the points $(0, 0)$ in the several planes E_t, $t \in R$.

We now consider an ensemble of complex two-dimensional spaces $C^2_{x,y}(t)$, $t \in Z$, where Z is the set of all complex numbers; let x_t, y_t be the complex coordinates of points in the spaces $C^2_{x,y}(t)$. Starting from these spaces we construct, via the equations (3.19) (with the x, y, s, t now complex numbers), a complex analytic manifold having no countable basis of open sets.

The complex dimension of the manifold so constructed is equal to two. By the same method we may construct complex manifolds without a countable basis of open sets and having arbitrary complex dimension $n > 1$.[1)]

1) See Calabi-Rosenlicht [1].

2. Non-compact complex manifolds on which all holomorphic functions are constants. We consider an example of sets of the indicated type due to E. Calabi and B. Eckmann [1]. We single out for attention the spaces C_z^{p+1} and $C_{z'}^{q+1}$, of the complex variables $z_0, z_1, \cdots, z_p$ and $z_0', z_1', \cdots, z_q'$, and the spheres

$$S = S_{2p+1} = \left\{ \sum_{s=0}^{p} |z_j|^2 = 1 \right\},$$

$$S' = S'_{2q+1} = \left\{ \sum_{k=0}^{q} |z_k'|^2 = 1 \right\}$$

in these spaces. The coordinates z_j $(j = 0, \cdots, p)$ and z_k' $(k = 0, \cdots, q)$ of the points $z \in S$ and $z' \in S'$ are taken as homogeneous coordinates of the points $w \in P^n$ and $w' \in P^q$ of the complex projective spaces $P^p = P$ and $P^q = P'$. Thus we define the mappings $\pi: S \longrightarrow P$ and $\pi': S' \longrightarrow P'$. For every point $z^0 \in S_{2p+1}$ the set $\pi^{-1}\pi(z^0)$ consists of all points of the sphere S_{2p+1} with coordinates $\lambda z_0^0, \lambda z_1^0, \cdots, \lambda z_p^0$, where $|\lambda| = 1$, and represents a great circle. Under the mapping π all points of this great circle go over into one and the same space P; moreover, we assert that under the mapping π the sphere S is stratified into the disks $\pi^{-1}\pi(z)$. Similarly, for every point $z' \in S'$ we define the great circle $\pi'^{-1}\pi'(z) \subset S'$. Under the mapping π' the sphere S' is stratified into the disks $\pi'^{-1}\pi'(z')$.

Next we consider the mapping $\Pi = \pi \times \pi' : S \times S' \longrightarrow P \times P'$ and the set $\Pi^{-1}\Pi(z^0, z^{0'})$ corresponding to all the points $(z^0, z^{0'}) \in S \times S'$. Each such set is a topological two-dimensional torus and consists of the points of the sets $S \times S'$ with coordinates $\lambda z_j^0, \lambda' z_k'^0$, where $j = 0, \cdots, p$, $k = 0, \cdots, q$, $|\lambda| = |\lambda'| = 1$. Under the mapping Π the product $S \times S'$ is stratified into these toruses.

Let $V_{\alpha\beta}$ $(\alpha = 0, \cdots, p;\ \beta = 0, \cdots, q)$ be an open subset of the product $S \times S'$, defined by the condition $z_\alpha z_\beta' \neq 0$. It is evident that $(p + 1)$ $(q + 1)$ open sets $V_{\alpha\beta}$ cover the product $S \times S'$, and the ensemble of open sets $\Pi(V_{\alpha\beta})$ covers the product $P \times P'$. In each open set $V_{\alpha\beta}$ we consider the quantities

$$_\alpha w_j = z_j z_\alpha^{-1}; \quad _\beta w_k' = z_k' z_\beta'^{-1}, \tag{3.20}$$

where $j = 0, \cdots, p$, $k = 0, \cdots, q$. For $j \neq \alpha$, $k \neq \beta$ these quantities are locally inhomogeneous coordinates in the open set $\Pi(V_{\alpha\beta})$.

Let τ (where $\operatorname{Im} \tau \neq 0$) be a complex constant; we consider for all points

$(z, z') \in V_{\alpha\beta}$ the quantities $\tau_{\alpha\beta}$ satisfying: 1) the equation

$$t_{\alpha\beta} \equiv \frac{1}{2\pi i}(\ln z_\alpha + \tau \ln z'_\beta)\ [\mathrm{mod}(1, \tau)]; \tag{3.21}$$

2) the condition $t_{\alpha\beta} \in T(1, \tau)$. Here $T(1, \tau) = T$ is the parallelogram constructed on the vectors 1 and τ; we adjoin to it its two non-parallel sides. Instead of the parallelogram T we shall later refer to the torus T obtained from the parallelogram by the identification of opposite sides; the complex numbers $t_{\alpha\beta}$ may be considered as coordinates of the points of this torus.

The functions ${}_\alpha w_j$, ${}_\beta w'_k$, $t_{\alpha\beta}$ define a differentiable mapping $\mu_{\alpha\beta}$: $V_{\alpha\beta} \longrightarrow C^{p+q} \times T$. Here C^{p+q} is the space of complex variables ${}_\alpha w_j$, ${}_\beta w'_k$ (where $j \neq \alpha$, $k \neq \beta$), and $C^{p+q} \times T$ is the complex manifold consisting of the product of this space and the torus T. We show that the mapping $\mu_{\alpha\beta}$ is homeomorphic.

We look on the quantities ${}_\alpha w_j$, ${}_\beta w'_k$ $(\alpha \neq j,\ k \neq \beta)$, $t_{\alpha\beta}$ as being given. It is easy to see that to them there corresponds a unique system z_j, z'_k $(j = 0, \cdots, p;\ k = 0, \cdots, q)$ satisfying equations (3.20), (3.21), and the condition

$$\sum_{j=0}^{p} |z_j|^2 = \sum_{k=0}^{q} |z'_k|^2 = 1; \quad z_\alpha,\ z'_\beta \neq 0. \tag{3.22}$$

In fact, we find from equations (3.20) and (3.22) that

$$z_j = {}_\alpha w_j z_\alpha, \quad z'_k = {}_\beta w'_k z'_\beta \quad (j \neq \alpha,\ k \neq \beta),$$

$$|z_\alpha| = \left[\sum_{j=0}^{p} |{}_\alpha w_j|^2\right]^{-\frac{1}{2}}, \quad |z'_\beta| = \left[\sum_{k=0}^{q} |{}_\beta w'_k|^2\right]^{-\frac{1}{2}}.$$

To determine the magnitudes of $\arg z_\alpha$ and $\arg z'_\beta$ we derive from equation (3.21) the equation

$$\begin{aligned} \arg z_\alpha + \tau \arg z'_\beta \equiv -i[(\ln z_\alpha + \tau \ln z'_\beta) - \\ - (\ln|z_\alpha| + \tau \ln|z'_\beta|)] \equiv 2\pi t_{\alpha\beta} + i(\ln|z_\alpha| + \\ + \tau \ln|z'_\beta|)\ [\mathrm{mod}(2\pi, 2\pi\tau)]. \end{aligned} \tag{3.23}$$

This equation uniquely defines $\arg z_\alpha$ and $\arg z'_\beta$ (since $\mathrm{Im}\,\tau \neq 0$) as differentiable functions of $t_{\alpha\beta}$, $|z_\alpha|$, and $|z'_\beta|$, which completes the proof that the mapping

$\mu_{\alpha\beta}$ is homeomorphic.

We look on the quantities ${}_\alpha w_j$, ${}_\beta w'_k$, $t_{\alpha\beta}$ (where $j \neq \alpha$, $k \neq \beta$) as local coordinates in the open set $V_{\alpha\beta}$. If $V_{\alpha\beta} \cap V_{\gamma\delta} \neq \emptyset$, the correspondence $\mu_{\alpha\beta}(V_{\alpha\beta} \cap V_{\gamma\delta}) \to \mu_{\gamma\delta}(V_{\alpha\beta} \cap V_{\gamma\delta})$, defined by the relationships

$$ {}_\gamma w_j = {}_\alpha w_j ({}_\alpha w_\gamma)^{-1}, \quad {}_\delta w'_k = {}_\beta w'_k ({}_\beta w'_\delta)^{-1} \tag{3.24}$$

(where $j = 0, \cdots, p$; $k = 0, \cdots, q$) and

$$ t_{\gamma\delta} \equiv t_{\alpha\beta} + \frac{1}{2\pi i} (\ln {}_\alpha w_\gamma + \tau \ln {}_\beta w'_\delta) \ [\operatorname{mod} (1, \tau)] \tag{3.25}$$

is a biholomorphic mapping.

Thus we have defined in the space $S \times S'$, by means of the mapping $\mu_{\alpha\beta} : V_{\alpha\beta} \to C^{p+q} \times T(1, \tau)$, the structure of a complex analytic manifold. We denote the complex manifold in question by $M_{p,q,\tau}$ or, more briefly, by $M_{p,q}$ when the value of τ is unimportant.

We choose the point $(z, z') \in V_{\alpha\beta}$ and the point $\Pi(z, z')$ corresponding to it under the mapping $\Pi : S \times S' \to P \times P'$. The quantities ${}_\alpha w_j$, ${}_\beta w'_k$ ($j \neq \alpha$, $k \neq \beta$) can be considered as (locally) inhomogeneous coordinates of the point $\Pi(z, z')$ in the open set $\Pi(V_{\alpha\beta})$, and the quantities ${}_\alpha w_j$, ${}_\beta w'_k$ ($j \neq \alpha$, $k \neq \beta$), $t_{\alpha\beta}$ are coordinates in the open set $V_{\alpha\beta}$. In these coordinates the mapping Π is written as the correspondence $({}_\alpha w_j, {}_\beta w'_k, t_{\alpha\beta}) \to ({}_\alpha w_j, {}_\beta w'_k)$; it is therefore a holomorphic mapping of the complex manifold $M_{p,q}$ on the space $P \times P'$. This mapping stratifies the manifold $M_{p,q}$ into fibers; the fiber F_ζ to which the point $\zeta({}_\alpha w_j^0, {}_\beta w_k'^0, t_{\alpha\beta}^0) \in M_{p,q}$ belongs is defined by the condition

$$ \Pi^{-1}\Pi({}_\alpha w_j^0, {}_\beta w_k'^0, t_{\alpha\beta}^0) = ({}_\alpha w_j^0, {}_\beta w_k'^0, T). \tag{3.26}$$

Calabi and Eckmann have established the following important property of the manifold $M_{p,q}$: Let $W \subset M_{p,q}$ be a compact irreducible analytic subset of this manifold, and let it have complex dimension $m < p + q + 1$. Then if the point $\zeta \in W$, the entire fiber (torus) $F_\zeta \subset W$. Let $f(\zeta)$, $\zeta \in M_{p,q}$ be a holomorphic (or even meromorphic) function on the manifold $M_{p,q}$. Then the complex $(p + q)$-dimensional analytic subset $\{f(\zeta) = \text{const}\}$ of the manifold $M_{p,q}$ consists of the torus F_ζ and the function $f(\zeta)$ that is constant on every such torus F_ζ. In other words, to every such function $f(\zeta)$ in the space $P \times P'$ there corresponds a holomorphic (meromorphic) function g such that $f(\zeta) = g(\Pi(\zeta))$.

Suppose that $a \in S$, $a' \in S'$ are arbitrary points of the spheres S and S'. Then the open sets $S \setminus \{a\}$, $S' \setminus \{a'\}$, $[S \setminus \{a\}] \times [S' \setminus \{a'\}]$ are homeomorphic to Euclidean spaces of the respective dimensions $2p + 1$, $2q + 1$ and $2p + 2q + 2$. The open set $[S \setminus \{a\}] \times [S' \setminus \{a'\}]$ with the complex analytic structure of the manifold $M_{p,q}$ is a non-compact complex manifold, which we denote by $E_{p,q}$. It is clear that $E_{p,q} \subset M_{p,q}$. Suppose that Π_0 is the restriction on $E_{p,q}$ of the mapping $\Pi: M_{p,q} \to P \times P'$.

If the point $(w, w') \in P \times P'$, the set $\Pi_0^{-1}(w, w') \subset E_{p,q}$ is: 1) a plane if $w = \pi(a)$, $w' = \pi'(a')$; 2) a cylinder if $w \neq \pi(a)$, $w' = \pi'(a')$; 3) a cylinder if $w = \pi(a)$, $w' \neq \pi'(a')$; 4) the torus F_ζ (where $(w, w') = \pi(\zeta)$) if $w \neq \pi(a)$, $w' \neq \pi'(a')$. The sets $\Pi_0^{-1}(w, w')$ of the last two types exist automatically if $q > 0$. In this case (for $q > 0$) the toruses $F_\zeta \subset E_{p,q}$ form an everywhere dense set in the manifold $E_{p,q}$.

We now assume that $f(\zeta)$, $\zeta \in E_{p,q}$ is a function holomorphic on the whole of the manifold $E_{p,q}$. This function is constant on the toruses F_ζ; it follows by considerations of continuity that it will also be constant on all the remaining sets $\Pi_0^{-1}(w, w')$ (of the first three types). Then $f(\zeta) = g(\Pi(\zeta))$, where the function g is holomorphic on the set $\Pi_0(E_{p,q})$, and accordingly on the whole space $P \times P'$. From this (by Liouville's theorem) it follows that the function g, and therefore the function f, are constants.

Thus, the ring of holomorphic functions on the complex manifold $E_{p,q}$ (homeomorphic to the topological $(2p + 2q + 2)$-dimensional Euclidean space) consists of constants.

3. **Holomorphically complete complex spaces.** The examples discussed in the preceding subsection show that to construct a meaningful theory of functions similar to the theory of holomorphic functions of one variable on non-compact Riemann surfaces, one must somehow restrict the class of complex spaces to be considered. We must single out a class of complex spaces that: 1) is similar in its properties to a non-compact Riemann surface; 2) admits sufficiently many holomorphic functions.

The holomorphically complete complex spaces form such a class. We proceed to describe their properties. We shall consider only normal complex spaces consisting of an at most countable set of connected components (see the end of §16 of this chapter).

DEFINITION (*F*-convex hull of a set). Let F be an ensemble of functions

holomorphic on the complex space R. By the F-convex hull $\hat{M}_F$ of a subset $M \subset R$ of the space R we shall mean the set of all points $r \in R$ for which $|f(r)| \leq \sup |f(M)|$ for all functions $f \in F$.

It is evident that the set $\hat{M}_F$ is always closed in R and contains the set M. If F is the ensemble of all functions holomorphic on the space R, then the set $\hat{M}_F$ is denoted by the symbol $\hat{M}$ and is called the holomorphically convex hull of the set M in the space R.

DEFINITION (holomorphically convex spaces). A complex space R is said to be holomorphically convex if the holomorphically convex hull $\hat{M}$ of every relatively compact set $M \subset R$ is compact.

It is easy to see that this definition is essentially the same as the corresponding definition of subsection 2, §11 concerning regions in the space P^n. We note again that the condition for holomorphic convexity can be formulated in still another way, as shown (after the relevant definition) in subsection 2, §11.

DEFINITION (holomorphically separable complex space). A complex space R is said to be holomorphically separable if for every two distinct points r_1, $r_2 \in R$ there exists a function $f(r)$, holomorphic in R, such that $f(r_1) \neq f(r_2)$.

DEFINITION (holomorphically complete complex space). A complex space R is said to be holomorphically complete if it is:

1) holomorphically convex;

2) K-complete, i.e., for every point $r_0 \in R$ there exist a neighborhood $U_{r_0} \subset R$ and functions, holomorphic on the space R,

$$(r) \qquad z_k = z_k(r), \quad k = 1, \cdots, n_{r_0}, \tag{3.27}$$

defining a nowhere exceptional mapping $r: U_{r_0} \longrightarrow C_z^{n_{r_0}}$. Here n_{r_0} is a positive integer.

The last requirement guarantees the existence of a sufficiently wide class of functions holomorphic on the complex space R.

In the particular case when R is a complex manifold enjoying the properties stated in the above definition, it is called a *holomorphically complete complex manifold.* Holomorphically complete complex manifolds are sometimes called Stein manifolds.[1)]

1) After K. Stein, who first studied them. See Stein [1].

Our definition applies to complex spaces of very general form. As we have already said, we limit ourselves to the case of normal complex spaces consisting of an at most countable set of connected components. For these we have the following theorem, due to H. Grauert [1].

THEOREM 18.1. *A holomorphically complete complex space always has a countable base of open sets.*

Thus Rado's classical theorem on Riemann surfaces extends to holomorphically complete complex spaces.

THEOREM 18.2. *A holomorphically complete complex space is always holomorphically separable.*

THEOREM 18.3. *For every point r_0 belonging to the holomorphically complete complex space R we may find a neighborhood $U_{r_0} \subset R$ and a set of functions*

$$(r) \qquad z_k = z_k(r), \quad k = 1, \cdots, n_{r_0}, \tag{3.28}$$

holomorphic in the space R and defining a one-to-one holomorphic mapping $r\colon U_{r_0} \to A$. Here $A \subset G \subset C_z^{n_{r_0}}$ is a normal analytic set in the region $G \subset C_z^{n_{r_0}}$.

We recall that an analytic set A is normal if the complex space $(A, \mathfrak{D}(A))$ is normal.

We note also that under a biholomorphic mapping of a complex space the properties of holomorphic convexity and completeness are preserved.

It should be remarked that in older works, published before the appearance of H. Grauert's paper (Grauert [1]) containing a proof of Theorems 18.1 – 18.3, the properties of holomorphically complete manifolds, as established in these theorems, were included in their definition (the notion of the holomorphically complete complex space was first formulated in Grauert's paper). For this reason, a holomorphically complete complex manifold was formerly defined [1)] as a holomorphically convex, holomorphically separable complex manifold $\mathfrak{M}$, having a countable basis of open sets and having the following property:

For every point $r_0 \in \mathfrak{M}$ there exists a system of functions holomorphic on the manifold $\mathfrak{M}$, serving as local coordinates in a neighborhood $U_{r_0} \subset \mathfrak{M}$ of that point.

R. Remmert has proved an imbedding theorem for holomorphically complete

1) See Cartan [5].

complex manifolds as follows.[1]

THEOREM 18.4. *Let $\mathfrak{M}$ be an n-dimensional holomorphically complete manifold. Then there exists a biholomorphic mapping $\pi: \mathfrak{M} \longrightarrow A$, where A is an analytic subset of the space C^N, and where $N = 2n + 1$, and A consists of ordinary points only.*

4. Examples of holomorphically complete complex manifolds. 1) For $n = 1$ every connected non-compact Riemann surface is a holomorphically complete complex manifold.[2]

2) Every region of holomorphy (without interior branches) of the space P^n, distinct from the whole space, is a holomorphically complete complex manifold. This conclusion follows from Theorem 11.4; the second requirement in the definition of a holomorphically complete complex space is clearly fulfilled in the present case.

3) The product $\mathfrak{M} \times \mathfrak{N}$ of the holomorphically complete complex manifolds $\mathfrak{M}$ and $\mathfrak{N}$ is itself a holomorphically complete complex manifold.

4) Let $\mathfrak{M}$ and $\mathfrak{N}$ be complex manifolds, with $\mathfrak{N} \subset \mathfrak{M}$, and let $\mathfrak{M}$ be holomorphically complete. Can we say under these circumstances that $\mathfrak{N}$ is also holomorphically complete?

H. Cartan [4] has formulated a sufficient condition: the complex manifold $\mathfrak{N}$ is holomorphically complete if it is *properly imbedded* in the manifold $\mathfrak{M}$.

This means that: a) the manifold $\mathfrak{N}$ is an analytic subset of the manifold $\mathfrak{M}$; b) if the point $\zeta \in \mathfrak{M}$, then in some neighborhood V_ζ (in the manifold $\mathfrak{M}$) we can introduce local coordinates with origin at the point ζ such that the set $\mathfrak{N} \cap V_\zeta$ is defined by the fact that at least one of these coordinates vanishes on it.

This condition allows us to assert, for example, that in the space C^n every closed algebraic manifold $\mathfrak{N}$ is holomorphically complete.

5) If $\mathfrak{M}$ is a holomorphically complete manifold, the set $\mathfrak{N} = (r \in \mathfrak{M}, f(r) \neq)\, 0 \subset \mathfrak{M}$, where f is a function holomorphic on the manifold $\mathfrak{M}$, is also a holomorphically complete manifold. This is easily seen in view of the fact that the function $1/f$ is holomorphic but not bounded on the set $\mathfrak{N}$.

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1) See R. Remmert, Dissertation, Münster, 1957. The equation $N = 2n + 1$ was established by Narasimhan [1].

2) This assertion follows from the results obtained in the Behnke-Stein paper [1].

An even stronger statement is valid. 1)

6) If $\mathfrak{M}$ is an n-dimensional holomorphically complete complex manifold and $A \subset \mathfrak{M}$ a purely $(n-1)$-dimensional analytic subset of it, then $\mathfrak{M} \setminus A$ is also a holomorphically complete complex manifold.

7) On the other hand, the complex projective space P^n, as a product of complex projective spaces, is not a holomorphically complete complex manifold. By Liouville's theorem, all holomorphic functions on this space reduce to constants, and it is therefore not holomorphically separable.

8) Similarly we may show that an arbitrary compact complex manifold is not holomorphically complete.

§19. RIEMANN DOMAINS

1. The concept of a Riemann domain. A domain of a covering of general type on the space C^n (a Riemann domain) is a generalization of the notion of plane domain of a covering, as considered in the second chapter.

DEFINITION (domain of a covering on the space C^n). The pair $\mathfrak{R} = (R, \Phi)$ is called a domain of a covering on the space C^n, or a Riemann domain if: 1) R is a Hausdorff space and Φ is a mapping of the space R on the space C^n; 2) to every point $r \in R$ there corresponds a neighborhood $U_r \subset R$, $V_{\Phi(r)} \subset C^n$, such that the triple $\mathfrak{U}_r = (U_r, \Phi, V_{\Phi(r)})$, is an analytic covering.

The point $\Phi(r) \in C^n$ is called the *fundamental point* of the point $r \in R$.

It follows from the present definition that Φ is a continuous nowhere exceptional mapping of the space R on the space C^n. Using the analytic covering $\mathfrak{U}_r$ we define on the space R a structural α-atlas, so that it becomes a normal complex space; the concepts of holomorphic function, holomorphic mapping, etc., retain their usual sense in it.

We remark that these concepts are often referred not to the complex space R, but to the corresponding Riemann domain $\mathfrak{R}$. For example, a function holomorphic relative to the complex space R is said to be holomorphic in the Riemann domain $\mathfrak{R}$ (or in suitable portions of this domain). In the sense of the structure we have defined, the mapping $\Phi: R \to C^n$ is obviously holomorphic

The Riemann domain $\mathfrak{R} = (R, \Phi)$ is said to be *non-ramified* (or locally

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1) See Docquier-Grauert [1].

one-sheeted) if the mapping Φ is locally homeomorphic. In the general case, the region $\mathfrak{R}$ is b-sheeted. Here b is equal to the maximum number of sheets of the covering U_r at the various points $r \in R$. Generally speaking, $1 \leq b \leq \infty$. If (R, Φ) is a Riemann domain, defined via the complex space R, the mapping $\Phi: R \longrightarrow C^n$ is said to be a *realization* of that space.

Many of the notions introduced in Chapter II for plane regions of a covering may be extended to the general case without essential change.

Thus, we shall say that a Riemann domain $\mathfrak{R} = (R, \Phi)$ is contained *within* the Riemann domain $\mathfrak{G} = (G, \Psi)$ and we will write $\mathfrak{R} < \mathfrak{G}$ if there exists a continuous mapping τ of the space R into the space G which conserves the fundamental points, i.e., which satisfies for all points $r \in R$ the equation $\Phi(r) = \Psi(\tau r)$.

If this mapping τ of the space R on the space $\tau R \subset G$ is a homeomorphism, the Riemann domain $\mathfrak{R}$ is called a *subregion* of the region $\mathfrak{G}$.

If $\tau R = G$, the Riemann domain $\mathfrak{R}$ is said to be *equivalent* to the region $\mathfrak{G}$, and we write $\mathfrak{R} = \mathfrak{G}$.

We define a *boundary point* r of the Riemann domain $\mathfrak{R} = (R, \Phi)$ as a filter r of connected regions $W \subset R$ satisfying the following properties: a) the regions $W \in r$ have no point in common belonging to the space R; b) the closures of the set $\{\Phi(W), W \in r\}$ have a unique common point $\zeta \in C^n$; and c) if $V_\zeta \subset C^n$ is a neighborhood of the point ζ, then one of the connected component sets $\Phi^{-1}(V_\zeta)$ belongs to the filter r; all regions $W \in r$ may be obtained in this way.

The ensemble of boundary points of a Riemann domain $\mathfrak{R}$ is called its boundary and is denoted by the symbol $\partial \breve{R}$. We write $\breve{R} = R \cup \partial \breve{R}$ and refer to $\breve{R}$ as the extended domain.

We consider the mapping $\breve{\Phi}: \breve{R} \longrightarrow C^n$, defined as follows:

$$\breve{\Phi}(r) = \Phi(r), \text{ if } r \in R; \quad \breve{\Phi}(r) = \zeta, \text{ if } r \in \partial R. \tag{3.29}$$

The mapping $\breve{\Phi}$ is said to be an extension of the mapping Φ on the extended domain $\breve{R}$.

We define a neighborhood W_{r_0} of the point $r_0 \in \partial \breve{R}$ in the set $\breve{R}$ as the union of a set $W_0 \in r_0$ with all filters $r \in \partial \breve{R}$, that contain at least one region $W \subset W_0$. Upon the introduction of these neighborhoods the extended domain $\breve{R}$ becomes a Hausdorff space, and $\breve{\Phi}: \breve{R} \longrightarrow C^n$ is a continuous mapping of it on

the space C^n.

We may then make the following statements:

1) The spaces R and $\breve{R}$ have a countable basis of open sets. [1)]

2) For every point $r_0 \in \partial\breve{R}$ it is possible to define a line $\breve{L} = \{r = r(t),$ $0 \leq t \leq 1\}$, $r(1) = r_0$, such that the line $\{r = r(t),\ 0 \leq t < 1\} \subset R$.

3) If the line $L = \{r = r(t),\ 0 \leq t < 1\} \subset R$, and the line $\Phi(L) \subset C^n$ can be continuously extended to an image of the closed interval $0 \leq t \leq 1$, then the line L can also be continuously extended, and moreover uniquely, to an image of the closed interval $0 \leq t \leq 1$.

This last statement is the basis for calling the boundary points $r \in \partial\breve{R}$ accessible.

2. **Holomorphy domains.** Let $\mathfrak{R}$, $\mathfrak{G}$ be Riemann domains, with $\mathfrak{R} < \mathfrak{G}$, and let τ be the mapping that sets up the correspondence. If the function p is holomorphic on the complex space R, and the function q is holomorphic on the space G, and if for all points $r \in R$ we have $p = q \circ \tau$, then the function q is called an *analytic* or *holomorphic extension* of the function p from the domain $\mathfrak{R}$ to the domain $\mathfrak{G}$.

A Riemann domain $\mathfrak{R}$ is called a domain of *holomorphy* if there exists in it a holomorphic function f which cannot be extended analytically to any Riemann domain $\mathfrak{G} > \mathfrak{R}$. In this case the domain $\mathfrak{R}$ is also called the domain of holomorphy (or of existence, or of regularity) of the given function f.

If the function q is an analytic extension of the function p from the Riemann domain $\mathfrak{R}$ to the domain $\mathfrak{G}$ and the latter is the domain of holomorphy of this function, the function q is said to be a *completely analytic function* and the function p is a *holomorphic functional element* of it.

The Riemann domain $\mathfrak{G}$ is called the *holomorphy hull* of the Riemann domain $\mathfrak{R}$ if all functions holomorphic in the domain $\mathfrak{R}$ can be extended to the domain $\mathfrak{G}$ but cannot be extended to any domain $\mathfrak{G}_1 > \mathfrak{G}$. In this case we write $\mathfrak{G} = H(\mathfrak{R})$.

A Riemann domain $\mathfrak{R} = (R, \Phi)$ is called *holomorphically convex* if the complex space R is holomorphically convex.

The notion of analytic convexity also applies in Riemann domains of general

1) See Grauert [1], Togari [1].

type. In formulating it, we use the concept of distinguished families of analytic sets. Such a family $G = \{G(w, t),\ 0 \le t \le 1\}$ is defined in the same way as in the case of a plane region of a covering (see subsection 8, §12, Chapter II), via the mapping $G(w, t)$ of the closed disk $|w| \le 1$ in the extended domain R. In the general case, however, we require that the mapping $\Phi \circ G(w, t)$ be holomorphic; in defining the family G_η we must require that the hypersphere of radius η in the space C^n contain the set $\bigcup_{0 \le t \le 1} [\Phi \circ G(w, t)]$.

The definition of an analytically convex domain, as formulated with the aid of distinguished families of analytic sets, goes over to the general case without any change whatever.

In H. Cartan's definition of analytic convexity it is now convenient to consider holomorphically complete manifolds rather than domains of holomorphy. 1)

We may assert theorems analogous to Theorems 11.4, 11.5, and 12.13. 2)

THEOREM 19.1. *A holomorphically convex Riemann domain is a domain of holomorphy.*

THEOREM 19.2. *A non-ramified Riemann domain of holomorphy is holomorphically convex.*

THEOREM 19.3. *A non-ramified Riemann domain is a domain of holomorphy if and only if it is analytically convex.*

The assertions of Theorems 19.2 and 19.3 do not hold for Riemann domains of general type. Noting this fact, Y. Togari [1] introduced another characteristic of Riemann domains of holomorphy.

DEFINITION (Γ-convex Riemann domain). The Riemann domain $\Re = (R, \Phi)$ is said to be Γ-convex if:

1) For every pair of points $r_1, r_2 \in R$, where $r_1 \neq r_2$ but $\Phi(r_1) = \Phi(r_2) = z \in C^n$, there exists in the space R a holomorphic function f defining at the point z different germs $f_{r_1} \circ \Phi^{-1}$ and $f_{r_2} \circ \Phi^{-1}$.

2) The holomorphically convex hull $\hat{W}$ of a compact set $W \subset R$ never belongs to the filter of domains defined by any (accessible) boundary point r of the domain R.

It is easy to see that a holomorphically convex Riemann domain is always

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1) See Docquier-Grauert [1].

2) See Grauert-Remmert [1].

Γ-convex; the converse is not in general true; nevertheless the following theorem (a strengthening of Theorem 19.1) holds:

THEOREM 19.4. *A Γ-convex Riemann domain is a domain of holomorphy.*

3. An example of a Riemann domain of holomorphy that is not holomorphically convex. [1] Let C_q^2, where $q = 1, \cdots, n-1$ and $n \geq 3$, be the space of the complex variables w_q, z_q. In the space $C^{2n-2} = \times_{q=1}^{n-1} C_q^2$, the product of the spaces C_q^2, we consider the cone $Y = \bigcup_{s \in P1} E_s$, where E_s is the analytic surface defined for $s \neq \infty$ by the equations $z_k = sw_k$ $(k = 1, \cdots, n-1)$, and for $s = \infty$ by the equations $w_k = 0$ $(k = 1, \cdots, n-1)$. Eliminating the parameter s from these equations we express the cone Y in the neighborhood of an arbitrary point $\eta \in Y$ as the ensemble of common zeros of a system of holomorphic functions. For example, if at the point $\eta \in Y$ all the coordinates $w_k \neq 0$ these equations have the form

$$(Y) \qquad \frac{z_1}{w_1} = \cdots = \frac{z_{n-1}}{w_{n-1}}. \tag{3.30}$$

Thus Y is an analytic set. All the points $\eta \in Y$, except the origin O, are ordinary points. In the neighborhood of an arbitrary such point the equations defining the cone Y can be solved in terms of any $n-2$ variables; therefore the set Y has at these points the complex dimension n. It has the same dimension at the origin O, since the point O is a cluster point of ordinary points. [2] Thus the set Y is pure-dimensional; we write $Y = Y^n$.

We shall show that the set Y^n is irreducible at the origin of coordinates. We must prove that whatever may be the radius of the sphere V with center at O, the set $Y^n \cap V$ cannot be split into two analytic sets Y_1 and Y_2 distinct from the set $Y^n \cap V$. In fact, under such a decomposition the sets Y_1 and Y_2 must have pure dimension n and consist, with the possible exception of the point O, of ordinary points only. Then the set $Y^n \cap (V \setminus O)$ must be disconnected, which is impossible, since

$$Y^n \cap (V \setminus O) = \bigcup_{s \in P1} [E_s \cap (V \setminus O)].$$

Thus, Y^n is a locally irreducible pure-dimensional analytic set in the

1) See Grauert-Remmert [1].

2) See Remmert-Stein [1], Theorem 13.

space C^{2n-2}. We can define a natural normal structure (see subsection 3, §17) on it, after which it becomes a normal complex space. It can be shown that the origin of coordinates is a (complex) non-uniformizable point of this space.

We consider the linear mapping $\Phi: Y^n \longrightarrow C_v^n$ defined by the equations

$$(\Phi) \qquad \left.\begin{aligned} v_0 &= z_1 + \cdots + z_{n-1}, \\ v_k &= w_k + e^{i\phi_k} z_k, \end{aligned}\right\} \tag{3.31}$$

where $k = 1, \cdots, n-1$, and the point $(w_1, z_1, \cdots, w_{n-1}, z_{n-1}) \in Y^n$, if $k \neq k'$. Here C_v^n is the space of the variables $v_0, v_1, \cdots, v_{n-1}$.

It is evident that the mapping Φ leaves the origin of coordinates fixed. It carries every plane $E_s \subset Y^n$, $s \in P^1$, into the $(n-1)$-dimensional analytic plane $\Phi(E_s) \subset C_v^n$, defined by the equations

$$\Phi(E_s) \qquad \omega(s, v) = v_0 \prod_{\kappa=1}^{n-1} (1 + e^{i\phi_\kappa} s) -$$

$$- s \sum_{\kappa=1}^{n-1} v_\kappa \prod_{\nu=1,\, \nu \neq \kappa}^{n-1} (1 + e^{i\phi_\nu} s) = 0. \tag{3.32}$$

It follows that the intersection $\Phi^{-1}(v) \cap E_s$, for an arbitrary point $v \in C_v^n$, reduces to at most a single point. The parameter s of a plane E_s for which the intersection is not empty satisfies the equation $\omega(s, v) = 0$ (if s is not infinite). The polynomial $\omega(s, v)$ is identically zero only for $v = 0$; for $v \neq 0$ its degree does not exceed $n-1$. There exist points $v \in C_v^n$, for which the equation $\omega(s, v) = 0$ has $n-1$ distinct roots s. This follows from the fact that the choice of an arbitrary point $v \in C_v^n$ corresponds to the arbitrary choice of $n-1$ coefficients of this equation (out of the total number n of coefficients).

Thus Φ is a holomorphic, proper, nowhere exceptional mapping. Every point $v \in C_v^n$ has $1 \leq q \leq n-1$ inverse images.

We now consider the set A' of points of the manifold $Y^n \setminus O$ at which the Jacobian of the mapping Φ vanishes. It is evident that A' is a purely $(n-1)$-dimensional analytic set in this manifold. It can be shown that since Φ is a proper, holomorphic, nowhere exceptional mapping of the manifold $Y^n \setminus O$ on the space $C_v^n \setminus O$, we have: 1) $A^* = \Phi(A')$ is a purely $(n-1)$-dimensional analytic set in the space $C_v^n \setminus O$; 2) the analytic set A^* can be extended to the

whole space C_v^n.[1)]

Let $A \subset C_v^n$ be the analytic set obtained as a result of this extension. It is evident that the point $O \in A$, and that the set $\Phi^{-1}(A) \subset Y^n$ does not split the space Y^n. Since $\Phi^{-1}(A) \subset [A' \cup O]$, the mapping Φ is locally homeomorphic on the set $Y^n \setminus \Phi^{-1}(A)$.

We have shown (see subsection 2, §14) that the triple (Y^n, Φ, C_v^n) defines an $(n-1)$-sheeted analytic covering on the space C_v^n.

Let $K = \{|s| < d < 1\}$ be a disk in the plane of the complex variable s. We consider the domain $R^n = \{\bigcup_{s \in K} E_s \setminus O\}$ and the Riemann domain $\mathfrak{R} = (R^n, \Phi)$ on the space C_v^n, where Φ is the mapping we are now studying.

We state a number of properties of the domain $\mathfrak{R}$.

I. Suppose $B^n = K \times (C_v^{n-1} \setminus O)$, where C_v^{n-1} is the space of the variables $v_1, \cdots, v_{n-1}$. The functions

$$(\pi) \qquad v_k = w_k, \quad s = \frac{z_q}{w_q} \tag{3.33}$$

(where $k, q = 1, \cdots, n-1$, except that the values of q for which $w_q = 0$ are excluded) define a biholomorphic mapping $\pi: R^n \to B^n$. It follows that R^n is a *complex manifold*.

II. We use the fact that the coefficients of the equation $\omega(s, v) = 0$ are determined by the choice of the point v. We choose the point $\mathfrak{w}_0 = (v_0^0, v_1^0, \cdots$ $\cdots, v_{n-1}^0) \in C_v^n$ so that: a) $\mathfrak{w}_0 \notin \Phi(E_\infty)$; b) the equation $\omega(s, \mathfrak{w}_0) = 0$ has only one simple root $s_0 \in K$ and has no other roots in the closed disk $\overline{K}$. Then there exists one and only one point $\mathfrak{b}_0 = \Phi^{-1}(\mathfrak{w}_0) \in E_{s_0} \setminus O \subset R_n$. By a continuity argument we may conclude that for all points $\mathfrak{w}$ in some neighborhood $U_{\mathfrak{m}_0} \subset C_v^n$ of the point $\mathfrak{w}_0$, each of the sets $\Phi^{-1}(\mathfrak{w})$ consists of one point. Thus we have shown that *the Riemann domain* $\mathfrak{R}$ *is one-sheeted at least at one point* $\mathfrak{w}_0 \in C_v^n$.

III. We adjoin to the domain R^n all its boundary points. Since R^n is a subdomain of the space $Y^n \setminus O$, they can be represented (we now leave out the origin of coordinates) as cluster points of the domain R^n that belong to the set $(Y^n \setminus O) \setminus R^n$. We extend the mapping π to the set $\breve{\partial} R \cap Y^n \setminus O$. Then every point $r \in [\breve{\partial} R^n \cap (Y^n \setminus O)]$ has a neighborhood $U_r \subset Y^n \setminus O$, in which the extended

1) See Remmert [1], Remmert-Stein [1].

mapping $\pi: U_r \to U^*_{r^*}$ is biholomorphic. Here $r^* = \pi r \in \partial K \times (C^{n-1}_v \setminus O)$, and $U^*_{r^*} \subset C^n_v$ is the corresponding neighborhood of the point r^*. Hence (by the definition of holomorphic mapping) we conclude:

If $f^*(s, v_1, \cdots, v_{n-1})$ is a function holomorphic in the domain B^n, having the point $r^* = \pi r$ (where $r \in \breve{\partial R^n} \cap (Y^n \setminus O)$) as a singular point, then the point $r \in \breve{\partial R^n}$ is a singular point of the function $f = f^* \circ \pi$.

IV. We prove that *over the point $O \in C^n_v$ there lies one and only one boundary point $r_0 \in \breve{\partial R^n}$*. It will be sufficient to show that the inverse image $\Phi^{-1}(V)$ of every sphere $V \subset C^n_v$ with center at the origin of coordinates is a connected non-empty subset of the domain R^n. But this is in fact the case, since

$$\pi[\Phi^{-1}(V)] = \{[s(v_1 + \cdots + v_{n-1}),\ v_1(1 + se^{i\phi_1}), \cdots$$
$$\cdots,\ v_{n-1}(1 + se^{i\phi_{n-1}})] \in V\} \cap B^n$$

is a connected subset of the domain B^n and $K \times O \subset \partial\pi[\Phi^{-1}(V)]$.

The continuum $K \times O$ of boundary points of the domain B^n corresponds to one and only one point $r_0 \in \breve{\partial R^n}$, while the points of the boundaries $\partial K \times (C^{n-1}_v \setminus O)$ and $\breve{\partial R^n} \setminus \Phi^{-1}(O)$ are in one-to-one correspondence. We can extend the mapping π^{-1} continuously to the set ∂B^n; let $\bar{\pi}^{-1}: \bar{B}^n \to \breve{R}^n$ be the mapping so obtained.

V. We now show that *the Riemann domain $\mathfrak{R}$ is a domain of holomorphy.* In fact, the polycylinder $K \times C^{n-1}_v$ is a domain of holomorphy for some function $f^*(s, v_1, \cdots, v_{n-1})$. Then in view of property III for the function $f = f^* \circ \pi$, holomorphic in the domain R^n, all points $r \in \breve{\partial R^n} \setminus r_0$ are singular points. The point r_0 is also a singular point of the function f, since it is a cluster point of the points $r \subset \breve{\partial R^n} \setminus r_0$.

We now assume that $\mathfrak{R}^* = (R^*, \Phi^*)$ is a domain of holomorphy of the function f and $R^n < R^*$. We extend the continuous mapping $\tau: R^n \to R^*$, preserving its fundamental point, to the space $\breve{R}^n$. As a result, we obtain the mapping $\breve{\tau}: \breve{R}^n \to \breve{R}^*$. This mapping carries none of the points $r \in \breve{\partial R^n}$ into the interior of the domain R^*, as all these points are singular points of the function $f(r)$. The mapping τ preserves the fundamental points, and in view of property II, does so in a one-to-one manner in some neighborhood of the point $\mathfrak{w}_0 \in C^n_v$. It follows that it is biunique in the whole domain R^n. Thus, $R^n = R^*$, and our assertion is proved.

VI. The domain $\mathfrak{R}$ *is not holomorphically convex.* In fact, for the holomorphy hull of the domain B^n we have $H(B^n) = K \times C_v^{n-1} \neq B^n$. Therefore the domain B^n is not holomorphically convex. But then neither is the Riemann domain $\mathfrak{R}$, since the property of holomorphic convexity is preserved under biholomorphic mappings (see subsection 3, §18).

We remark that, as Y. Togari proved [1], the non-holomorphically-convex holomorphy domain we have constructed is Γ-convex. Up to now we know of no examples of holomorphy domains that are not Γ-convex.

The idea underlying the method of constructing a Riemann domain of holomorphy $\mathfrak{R} = (R_n, \Phi)$ that is not holomorphically convex may be used in the construction of a broad class of such domains. [1)]

1) See Scheja [1].

CHAPTER IV

INTEGRAL REPRESENTATIONS

§20. THE FUNDAMENTAL THEOREM OF CAUCHY-POINCARÉ. THEORY OF RESIDUES ON A COMPLEX MANIFOLD

1. The integral over a manifold. In §1, Chapter I we defined an integral over a k-dimensional, l-smooth manifold ($l \geq 1$) consisting of a single element. We now consider integrals taken over manifolds of a more general type. We shall make use of an auxiliary concept, i.e., that of the partition of unity (Dieudonné's partition).

Let $\mathcal{V}$ be a piece-wise l-smooth ($l \geq 1$) topological p-dimensional manifold having a countable basis of open sets. We single out from its structural atlas a countable, or if possible a finite, set of charts (V_j, ψ_j), $j = 1, 2, \cdots$, satisfying the condition $\bigcup V_j = \mathcal{V}$. The family of functions $\lambda_j(x)$, $j = 1, 2, \cdots$, defined on the manifold $\mathcal{V}$, is a partition of unity, subordinate to the covering $\{V_j\}$, if: 1) the functions $\lambda_j(x)$, $j = 1, 2, \cdots$, belong to the class $\mathcal{C}^l$ on every l-smooth component of the manifold $\mathcal{V}$; 2) $0 \leq \lambda_j(x) \leq 1$, $j = 1, 2, \cdots$, and $\sum_{j=1}^{\infty} \lambda_j(x) = 1$ at all points $x \in \mathcal{V}$; 3) every function $\lambda_j(x)$ has a compact carrier contained in the domain V_j; 4) every point $x_0 \in \mathcal{V}$ has a neighborhood intersecting with at most a finite set of the carriers of the function $\lambda_j(x)$.[1]

It is evident from Borel's covering lemma that an arbitrary compact set $K \subset \mathcal{V}$ intersects with at most a finite number of carriers of the functions $\lambda_j(x)$.

1) For a proof of the existence (an effective construction) of a family of functions constituting a partition of unity, see for example G. de Rham, *Variétés differentiables*, Actualités Sci. Ind. No. 1222, Hermann, Paris, 1955; Russian transl., IL, Moscow, 1956; pp. 21–26.

If α is an exterior differential form[1] of class $\mathcal{C}^1$ and of degree p, defined on the manifold $\mathcal{V}$, and its carrier is included in a single domain $\psi_j V_j$, its integral over the manifold $\mathcal{V}$ is defined by equations (1.15_1) and (1.15_2). When the form α has an arbitrary carrier, we write

$$\int_{\mathcal{V}} \alpha = \sum_{j=1}^{\infty} \int_{\mathcal{V}} \lambda_j \alpha, \tag{4.1}$$

where $\{\lambda_j(x),\ j = 1, 2, \cdots\}$ is the partition of unity described above. Every term of the series (4.1) can be reduced, in view of the properties of the functions $\lambda_j(x)$, to an integral of the type of (1.15_1) or (1.15_2). If the series converges for an arbitrary partition of unity, subject to the properties already listed, then equation (4.1) defines the integral $\int_{\mathcal{V}} \alpha$; in this case one may also say that the integral $\int_{\mathcal{V}} \alpha$ converges. Clearly, the value of the integral (4.1) does not in this case depend on the choice of the partition of unity by which it is constructed.

If the carrier of the form α is compact, the integral (4.1) necessarily converges.

We consider on the manifold $\mathcal{V}$ a smooth p-dimensional element of the chain $v = (w, \mu)$. Here (see subsection 6 of the Introduction) w is a simplex in the space R_p of the real variables $t_1, \cdots, t_p$, oriented in the same way as the space R_p by the coordinate system $t_1, \cdots, t_p$; μ: $\overline{w} \longrightarrow \mathcal{V}$ is a smooth mapping of the closed simplex $\overline{w}$ on the manifold $\mathcal{V}$.

If the carrier of the form α is contained in a single domain $V_j \supset \mu w$, we define the integral of the form α over the element of the chain v by the equation

$$\int_v \alpha = \int_w \alpha \circ \mu \tag{4.2_1}$$

(if α is an odd form), and by

$$\int_v \alpha = \epsilon \int_w \alpha \circ \mu \tag{4.2_2}$$

(if α is an even form; here ϵ is the corresponding orientation). The equations (4.2_1) and (4.2_2), in the particular case when μ is a homeomorphism, reduce to

1) In this chapter we will consider only such forms and for brevity we will simply call them "forms."

the equations (1.15_1) and (1.15_2). When the form α has an arbitrary carrier the integral $\int_v \alpha$ must be defined by recourse to the series (4.1). The integral of the form α over the chain $V = \Sigma c_k v_k$ is defined by the equation

$$\int_V \alpha = \Sigma c_k \int_{v_k} \alpha. \tag{4.3}$$

Here the c_k are arbitrary complex numbers.

We noted in the Introduction that one and the same chain may be specified in a number of ways as a linear form in terms of its elements. A criterion that such linear forms define one and the same chain is that on them the integrals (4.3) coincide for all forms α.

2. The theorems of Stokes and Cauchy-Poincaré. Let $V \subset \mathcal{U}$ be a topologically p-dimensional chain consisting of smooth elements, and let ϕ be a form of class $\mathcal{C}^1$ and of degree $p-1$. Then we have the equation[1)]

$$\int_V d\alpha = \int_{\partial V} \alpha. \tag{4.4}$$

This is Stokes' theorem. It includes as particular cases the classical formulas of Newton-Leibnitz, Green, Stokes and Gauss-Ostrogradskiĭ. The general case (with somewhat different notation) was first taken up by H. Poincaré.

Stokes' theorem (4.4) implies that for a closed form α

$$\int_{\partial V} \alpha = 0; \tag{4.5}$$

and for an arbitrary form $\alpha \in \mathcal{C}^1$ and arbitrary cycle V we have

$$\int_V d\alpha = 0. \tag{4.6}$$

This equation contains the classical statement that the integral of the differential of any function vanishes when taken around a closed contour.

Let us now consider a holomorphic function ϕ on the complex analytic manifold Z^n, given on each of its charts by an expression of the form

$$\phi = f(z)\, dz, \tag{4.7}$$

1) See G. de Rham, op. cit.

where $z_1, \cdots, z_n$ are locally complex coordinates, $dz = dz_1 \wedge \cdots \wedge dz_n$, and the function $f(z)$ is holomorphic on Z^n. It is clear that $d\phi = 0$ and therefore the form ϕ is closed. It follows that for an arbitrary topologically $(n+1)$-dimensional chain $V \subset Z^n$, consisting of smooth elements, we have by (4.5)

$$\int_{\partial V} f(z)\, dz = 0, \tag{4.8}$$

where again $dz = dz_1 \wedge \cdots \wedge dz_n$. This equation amounts to a statement of the following theorem.

THEOREM 20.1 (fundamental theorem of Cauchy-Poincaré). *If the function $f(z)$ is holomorphic on the complex manifold Z^n, and for the topologically $(n+1)$-dimensional chain V we have $V \subset Z^n$, then equation* (4.8) *holds.*

Theorem 20.1 holds also for chains of a more general type (the elements need not be smooth).

3. Form-residue on a complex manifold.[1] We consider a complex manifold $S^{(i)} = (S^{n-1})^{(i)}$, $\Sigma^{(j)} = (\Sigma^{n-1})^{(j)}$ $(i = 1, \cdots, m,\ j = 1, \cdots, m')$, which is analytically locally regularly imbedded in the complex manifold $Z = Z^n$. Let these manifolds be defined in the neighborhoods $U_{z^{(0)}}$ of their points $z^{(0)}$ by the equations (0.1_4), which have the form

$$\left.\begin{aligned} &(S^{(i)}) \quad s_i(z|z^{(0)}) = 0,\ i = 1, \cdots, m, \\ &(\Sigma^{(j)}) \quad \sigma_j(z|z^{(0)}) = 0,\ j = 1, \cdots, m'. \end{aligned}\right\} \tag{4.9}$$

Here s_i, σ_j are holomorphic functions on the manifold Z at the points $z^{(0)}$ and grad $s_i \neq 0$. The manifold $S^{(i)}$ consists entirely of ordinary points. We now assume that the manifolds $S^{(i)}$, $\Sigma^{(j)}$ lie in general positions in the manifold Z, i.e., that for all their ordinary points the matrices composed of the derivatives of the functions (4.9) have, in local coordinates, the highest possible rank.

Using the manifolds (4.9) as a basis, we form the manifold

$$S = S^{(1)} \cap \cdots \cap S^{(m)};\ \Sigma = \Sigma^{(1)} \cup \cdots \cup \Sigma^{(m')}. \tag{4.10}$$

Let ϕ be a form of degree $0 < l \leq 2n$, vanishing on the manifold Σ, belonging to

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1) See Leray [1].

the class $\mathcal{C}^\infty$ on the manifold $Z\setminus(S^{(1)} \cup \cdots \cup S^{(m)})$, having $S^{(1)}, \cdots, S^{(m)}$ as its polar manifolds with respect to the sequence $p_1, \cdots, p_m$ (where $p_1, \cdots$ $\cdots, p_m$ are natural numbers). This last requirement means that in some neighborhood $U_{z^{(0)}}$ of every point $z^{(0)} \in S^{(i_1)} \cap \cdots \cap S^{(i_r)}$, where $1 \leq i_1 < \cdots$ $\cdots < i_r \leq m$, the form

$$[s_{i_1}(z|z^{(0)})]^{p_{i_1}} \ldots [s_{i_r}(z|z^{(0)})]^{p_{i_r}} \phi \tag{4.11}$$

belongs to the class $\mathcal{C}^\infty$, but no longer belongs to that class if any one of the quantities $p_1, \cdots, p_m$ is decreased. By the value of the form (4.11) at the points $z \in S^{(i)}$, $i = 1, \cdots, m$, we here mean its continuous extension at these points.

We first take $m = 1$ and write S instead of $S^{(1)}$ and s instead of s_1. In this subsection we limit ourselves to polar manifolds of order one. Polar manifolds of higher order will be considered later.

We have

THEOREM 20.2. *Let $\phi \in \mathcal{C}^\infty$ be a closed form on the manifold $Z\setminus S$ and let S have a polar manifold of order one.*

Then in some neighborhood $U_{z^{(0)}}$ of every point $z^{(0)} \in S$ there exist forms $\psi, \theta \in \mathcal{C}^\infty$, such that for $z \in U_{z^{(0)}}\setminus S$

$$\phi = \frac{ds\,(z|z^{(0)})}{s\,(z|z^{(0)})} \wedge \psi(z|z^{(0)}) + \theta(z|\,z^{(0)}). \tag{4.12}$$

The trace $\psi(z|z^{(0)})|_S$ depends only on the form ϕ and is a closed form.

If the form ϕ is holomorphic, the trace $\psi(z|z^{(0)})|_S$ is holomorphic at the point $z^{(0)}$.

The proof of Theorem 20.2 is based on the following lemma.

LEMMA. *Let $\alpha \in \mathcal{C}^\infty$ be a form on the manifold Z. Then it can be represented in the neighborhood $U_{z^{(0)}}$ of any point $z^{(0)} \in S$ as follows:*

$$\alpha = ds \wedge \beta \tag{4.13}$$

(*where the form $\beta \in \mathcal{C}^\infty$ for $z \in U_{z^{(0)}}$*), *if and only if*

$$ds \wedge \alpha = 0. \tag{4.14}$$

Here the trace $\beta|_S$ *of the form is uniquely defined; if the form* α *is holomorphic, the trace* $\beta|_S$ *is holomorphic at the point* $z^{(0)}$.

The truth of this lemma is obvious if we choose the function $s(z|z^{(0)})$ as one of the local coordinates on the manifold Z in a neighborhood of the point $z^{(0)}$, as is permissible, since $z^{(0)}$ is an ordinary point of the manifold S.

We now pass directly to the proof of Theorem 20.2. In some neighborhood $U_{z^{(0)}}$ of the point $z^{(0)}$ the form $s\phi \in \mathcal{C}^\infty$, and therefore $d(s\phi) \in \mathcal{C}^\infty$. For $z \in U_{z^{(0)}} \setminus S$ we have

$$d(s\phi) = ds \wedge \phi \tag{4.15}$$

(since then $d\phi = 0$). This equation permits us to extend the form $ds \wedge \phi$ to the point $z \in U_{z^{(0)}} \cap S$. After the extension, the form $ds \wedge \phi \in \mathcal{C}^\infty$ everywhere in $U_{z^{(0)}}$. It follows at once from (4.15) that the forms $\alpha = ds \wedge \phi$, in the neighborhood $U_{z^{(0)}}$, satisfy (4.14). Our lemma, applied to the neighborhood $U_{z^{(0)}}$, proves that in some neighborhood of the point $z^{(0)}$ there exists a form $\theta = \theta(z|z^{(0)}) \in \mathcal{C}^\infty$, such that

$$ds \wedge \phi = ds \wedge \theta. \tag{4.16}$$

Multiplying both sides of equation (4.16) by s and taking into account the fact that $s\phi$, $s\theta \in \mathcal{C}^\infty$ in the given neighborhood of the point $z^{(0)}$, we find that $s\phi - s\theta \in \mathcal{C}^\infty$ and

$$ds \wedge (s\phi - s\theta) = 0. \tag{4.17}$$

Then by the lemma there exists in some neighborhood of the point $z^{(0)}$ a form $\psi = \psi(z|z^{(0)}) \in \mathcal{C}^\infty$, such that

$$s\phi - s\theta = ds \wedge \psi, \tag{4.18}$$

$$\phi = \frac{ds}{s} \wedge \psi + \theta. \tag{4.12}$$

The second equation holds, of course, outside of the manifold S.

It is easy to see that if ϕ is holomorphic on the manifold $Z \setminus S$, the forms ψ and θ are also holomorphic.

We now show that the trace $\psi|_S$ is defined by the choice of the form ϕ and the function s. It is sufficient to show that the equation

$$0=\frac{ds}{s}\wedge\psi+\theta \tag{4.12'}$$

implies that

$$\psi|_S=0. \tag{4.19}$$

In fact, it follows from (4.12) that

$$ds\wedge\psi+s\theta=0. \tag{4.12''}$$

Multiplying by ds we obtain $ds\wedge s\theta=0$. Here s is a scalar, and therefore we have in $U_{z^{(0)}}\backslash S$ the equation $ds\wedge\theta=0$. Since the form $ds\wedge\theta$ belongs to the class $\mathcal{C}^\infty$, this equation holds throughout the neighborhood $U_{z^{(0)}}$. Therefore, still in view of our lemma, we find that in some neighborhood of the point $z^{(0)}$ there exists a form $\omega\in\mathcal{C}^\infty$, such that $\theta=ds\wedge\omega$. When we substitute this expression for the form θ in equation (4.12′) we find that $ds\wedge(\psi+s\omega)=0$. We apply our lemma again and see that in some neighborhood of the point $z^{(0)}$

$$\psi+s\omega=ds\wedge\tilde{\omega},$$

where $\tilde{\omega}\in\mathcal{C}^\infty$. Finally, noting that ψ is a form of degree $l>0$, we arrive at equation (4.19).

We now show that the trace $\psi|_S$ is also independent of the choice of the function $s(z|z^{(0)})$ used in defining the manifold S in the neighborhood of the point $z^{(0)}$ and is therefore completely defined by the form ϕ.

In fact, if the manifold S can be defined in some neighborhood of the point $z^{(0)}$ by a holomorphic mapping $s^*(z|z^{(0)})=0$, where ss^{*-1} and s^*s^{-1} are holomorphic in that neighborhood, then

$$\phi=\frac{ds}{s}\wedge\psi+\theta=\frac{ds^*}{s^*}\wedge\psi+\left[\frac{ds}{s}-\frac{ds^*}{s^*}\right]\wedge\psi+\theta=\frac{ds^*}{s^*}\wedge\psi+\theta^*,$$

where

$$\theta^*=\left[d\ln\frac{s}{s^*}\right]\wedge\psi+\theta\in\mathcal{C}^\infty$$

in the selected neighborhood of the point $z^{(0)}$. Our assertion follows.

We have still to show that the form $\psi|_S$ is closed. We conclude from (4.12) that

$$-\frac{ds}{s}\wedge d\psi+d\theta=0.$$

This equation is the analogue of (4.12′) with respect to the forms $d\psi$ and $d\theta$. We know that equation (4.19) follows from it. In the present case, this means that $d\psi|_S = 0$.

DEFINITION (form-residue). The trace $\psi(z|z^{(0)})|_S$ of the form emerging from the representation (4.12) is called the form-residue of the form ϕ on the first-order polar manifold S in the neighborhood of the point $z^{(0)} \in S$.

We introduce the notation:

$$\operatorname{res}[\phi] = \frac{s\phi}{ds}\bigg|_S = \psi(z|z^{(0)})|_S. \tag{4.20}$$

4. Properties of the form-residue. To begin with, we take note of certain special cases of formula (4.20). If in the neighborhood $U_{z^{(0)}}$ of any point $z^{(0)} \in S$ we have

$$\phi = \frac{\omega(z|z^{(0)})}{s(z|z^{(0)})}, \tag{4.21}$$

where the form $\omega(z|z^{(0)}) \in \mathcal{C}^\infty$ for $z \in U_{z^{(0)}}$, we shall write, instead of (4.20),

$$\operatorname{res}[\phi] = \frac{\omega}{ds}\bigg|_S = \psi(z|z^{(0)}). \tag{4.20$_1$}$$

If the form ω is closed, then since $d\omega = 0$ and $d\phi = 0$ we find from (4.21) that

$$ds \wedge \omega = 0. \tag{4.21$_1$}$$

In the particular case when

$$\omega(z|z^{(0)}) = f(z)\, dz,$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$, when $f(z) \in \mathcal{C}^\infty$ in a neighborhood $\tilde{S}$ of the manifold S, and $z_1, \cdots, z_n$ are local coordinates in the neighborhood $U_{z^{(0)}}$, the form-residue $\operatorname{res}[\phi]$ can be calculated by the formula

$$\operatorname{res}[\phi] = (-1)^{j-1} f(z)\, \frac{dz_1 \wedge \cdots \wedge [dz_j] \wedge \cdots \wedge dz_n}{s_{z_j}} \tag{4.20$_2$}$$

(for $s_{z_j} \neq 0$). Here, as always, the symbol $[dz_j]$ indicates that the differential dz_j is omitted from the list of factors. We can prove the validity of (4.20$_2$) by representing the form ϕ as

$$\phi = \frac{f}{s}(-1)^{j-1}\frac{1}{s_{z_j}}(s_{z_j}dz_j)\wedge(dz_1\wedge\cdots\wedge[dz_j]\wedge\cdots\wedge dz_n) =$$

$$= \frac{ds}{s}\wedge\frac{(-1)^{j-1} f dz_1\wedge\cdots\wedge[dz_j]\wedge\cdots\wedge dz_n}{s_{z_j}}.$$

We now consider the case when the polar manifold S of the form ϕ is given in the large by an equation $s(z) = 0$, where the function $s(z)$ is holomorphic in some neighborhood $\tilde{S}$ of the manifold S and satisfies in the neighborhood of every point $z^{(0)} \in S$ all the conditions stated above. Then, by Theorem 20.2, for $z \in U_{z^{(0)}}$, we have

$$\phi = \frac{ds(z)}{s(z)}\wedge\psi(z|z^{(0)}) + \theta(z|z^{(0)}). \tag{4.12$_1$}$$

We assume that there exists on the manifold Z a partition of unity $\{\lambda_i(z),\ i = 1, 2, \cdots\}$. Then, from (4.12$_1$) we obtain the equation

$$\phi(z) = \frac{ds(z)}{s(z)}\wedge\psi(z) + \theta(z), \tag{4.12$_2$}$$

where

$$\psi(z) = \sum_j \lambda_j(z)\psi(z|z^{(j)}),\quad \theta(z) = \sum \lambda_j(z)\theta(z|z^{(j)}).$$

Here it is assumed that the neighborhoods of the points $z^{(j)} \in S$, $j = 1, 2, \cdots$, cover the entire manifold S.

We define the form-residue $\operatorname{res}[\phi]$ of the form ϕ on the polar first-order manifold S as the trace $\psi(z)|_S$.

From the definition of the form-residue $\operatorname{res}[\phi]$ of the form ϕ it follows that

$$\operatorname{res}[\phi]|_{\Sigma\cap S} = 0. \tag{4.22}$$

We now consider an operator δ which associates to every point $z^{(0)} \in S$ a homeomorphic image of its periphery $\delta z^{(0)} \subset Z\backslash S$ such that the following conditions are satisfied:

a) In some neighborhood $U_{z^{(0)}} \subset Z$ there exists a system of coordinates $z_1, \cdots, z_n$ with origin at the point $z^{(0)}$ and such that the manifold $S \cup U_{z^{(0)}}$ is defined by the equation $z_n = 0$, and the line $\delta z^{(0)} \subset U_{z^{(0)}}$ is defined by the equation $|z_n| = 1$.

b) The lines $\{\delta z,\ z \in S\}$ form a continuous family in the manifold $Z \backslash S$.

c) For $z^{(1)} \neq z^{(2)}$ the lines $\delta z^{(1)}$ and $\delta z^{(2)}$ have no point in common.

d) If $z \in \Sigma$, then $\delta z \subset \Sigma$.

In their totality, the lines $\{\delta z,\ z \in S\}$ form the boundary of the "tubular" neighborhood of the submanifold S in the manifold Z. The construction is possible, since the manifold S consists of ordinary points and the topological dimensions of the manifolds S and Z differ by two units.

Property a) permits us to introduce an orientation on the closed line $\delta z^{(0)}$. Let the orientation of the manifold Z be given by the order of the sequence $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \cdots, n$, and that of the manifold S by the order of the sequence $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \cdots, n-1$. Then the positive orientation of the line $\delta z^{(0)}$ is to correspond to a circuit in the order given by the sequence of points $z_n = 1$, $z_n = i$, $z_n = -1$.

Let γ be a cycle on the manifold S, compact relative to the manifold Σ. We define $\delta\gamma$ as the union of the lines δz for all points $z \in \gamma$. It is easy to see that $\delta\gamma \subset Z\backslash S$ is a cycle which is compact relative to the manifold Σ. The topological dimension of the cycle $\delta\gamma$ is greater by one unit than the topological dimension of the cycle γ. The cycle $\delta\gamma$ is stratified into the lines $\delta(z)$, $z \in \gamma$; the cycle γ is a base for this stratification. Since the lines δz, $z \in \gamma$ are oriented, we may determine the orientation of the cycle $\delta\gamma$ in terms of the orientation of the cycle γ.

Let $H_c(S, \Sigma)$ and $H_c(Z\backslash S, \Sigma)$ be homology groups of the manifolds S and $Z\backslash S$ with compact carriers relative to the manifold Σ_g and with integer coefficients. The operator δ generates a homomorphism between these homology groups:

$$\delta\colon H_c(S, \Sigma) \longrightarrow H_c(Z\backslash S, \Sigma). \tag{4.23}$$

We have the following

THEOREM 20.3 (Leray [1]). *Let the cycle γ belong to the homology class $h_c(S, \Sigma) \in H_c(S, \Sigma)$ and let its dimension satisfy $D(\gamma) = l - 1$. Then for a closed form ϕ having S as its first-order polar manifold we have*

$$\int_{\delta\gamma} \phi = 2\pi i \int_{\gamma} \operatorname{res}[\phi]. \tag{4.24}$$

Since γ is an arbitrary cycle belonging to the homology class $h_c(S, \Sigma)$, formula (4.24) may usefully be written in the form

$$\int_{\delta h_c(S,\,\Sigma)} \phi = 2\pi i \int_{h_c(S,\,\Sigma)} \operatorname{res}[\phi]. \tag{4.24$_1$}$$

5. **Class-residue on a complex manifold.** We denote by $(\widetilde{Z}, \Sigma)$ an ensemble of forms belonging to the class $\mathcal{C}^\infty$ on some manifold $\widetilde{Z}$ and vanishing on the submanifold $\Sigma \subset \widetilde{Z}$, and we denote by $H^*(\widetilde{Z}, \Sigma)$ the homology ring[1)] of closed forms belonging to this ensemble (see subsection 8 of the Introduction). We have

THEOREM 20.4. *Let there be given a closed form $\phi \in h^*(Z\setminus S, \Sigma)$. Then in the class $h^*(Z\setminus S, \Sigma)$ there exists a form χ, having S as its first-order polar manifold. The ensemble of form-residues on the manifold S for these forms χ is a relative cohomology class for the pair (S, Σ).*

DEFINITION (class-residue). The cohomology class of form-residues defined by the closed form $\phi \in h^*(Z\setminus S, \Sigma)$ on the manifold S is called the *class-residue*, and is denoted by the symbol

$$\operatorname{Res}[\phi] = \operatorname{Res}[h^*(Z\setminus S, \Sigma)].$$

Clearly, if $\operatorname{res}[\phi]$ exists, then $\operatorname{res}[\phi] \in \operatorname{Res}[\phi]$.

The construction of the class-residue $\operatorname{Res}[\phi]$ defines, in view of Theorem 20.4, a homomorphic mapping of the cohomology ring $H^*(Z\setminus S, \Sigma)$ on the cohomology ring $H^*(S, \Sigma)$

$$\operatorname{Res}\colon\ H^*(Z\setminus S, \Sigma) \longrightarrow H^*(S, \Sigma). \tag{4.25}$$

The cohomology rings $H^*(Z\setminus S, \Sigma)$ and $H^*(S, \Sigma)$ are algebras over the cohomology ring $H^*(Z)$, since after multiplication by the class $h^*(Z) \in H^*(Z)$ the classes $h^*(S, \Sigma) \in H^*(S, \Sigma)$ and $h^*(Z\setminus S, \Sigma) \in H^*(Z\setminus S, \Sigma)$ remain members as before of these cohomology rings (multiplication is carried out by the rule given in subsection 8 of the Introduction). The operation of forming the class-residue defines a homomorphism of the algebra $H^*(Z\setminus S, \Sigma)$ since

$$\operatorname{Res}[h^*(Z\setminus S, \Sigma)\, h^*(Z)] = \{\operatorname{Res}[h^*(Z\setminus S, \Sigma)]\}\, h^*(Z).$$

To Theorem 20.3 there corresponds in the general case the following theorem.

THEOREM 20.5. *Let the cycle γ belong to the homology class $h_c(S, \Sigma)$*

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1) It is known that this ring coincides with the cohomology ring of the manifold $\widetilde{Z}$ relative to the manifold Σ with an arbitrary carrier and complex coefficients. See G. de Rham, *Variétés différentiables*, Actualités Sci. Ind. No. 1222, Hermann, Paris, 1955; Russian transl., IL, Moscow, 1956; Chapter IV.

and have dimension $D(\gamma) = l - 1$. *Then we have for the closed form* $\phi \in \mathcal{C}^\infty$ *on the manifold* $Z \setminus S$

$$\int_{\delta\gamma \in \delta h_c(S, \Sigma)} \phi = 2\pi i \int_{\gamma \in h_c(S, \Sigma)} \operatorname{Res}[\phi]. \tag{4.26}$$

The integral on the right-hand side of equation (4.26) *is taken for an arbitrary element of the class-residue* Res$[\phi]$ *and does not depend on the choice of this element.*

We note that in Theorem 20.5 it is impossible to limit ourselves to consideration of holomorphic forms. Even if the form ϕ is holomorphic, the class-residue Res$[\phi]$ may be devoid of holomorphic forms.

For $l = 1$, $n = 1$ the class-residue Res$[f(z)\,dz]$ reduces to the ordinary residue of the function $f(z)$ at its singular points.

We now consider the general case $m > 1$, when the form ϕ has polar manifolds $S^{(1)}, \cdots, S^{(m)}$ of orders $p_1, \cdots, p_m$. We construct the sequence of homomorphisms

$$\delta^m\colon H_c(S_m, \Sigma) \to \cdots \to H_c(S_i, \Sigma) \to H_c(S_{i-1}, \Sigma) \to \cdots$$
$$\cdots \to H_c(S_1, \Sigma) \to H_c(S_0, \Sigma), \tag{4.27}$$

where $S_i = S^{(1)} \cap \cdots \cap S^{(i)} \setminus S^{(i+1)} \cup \cdots \cup S^m$, $i = 1, \cdots, m$, $S_m = S$ and $S_0 = Z \setminus S^{(1)} \cup \cdots \cup S^{(m)}$. These homomorphisms establish a correspondence between the cycles $\gamma \in h_c(S_i, \Sigma)$ and the cycles $\delta\gamma \in h_c(S_{i-1}, \Sigma)$. The cycle $\delta\gamma$ is stratified into closed lines (homeomorphic to circles) traversing the manifold S_i once and belonging to the manifold S_{i-1}.

We then construct the sequence of homomorphisms

$$\mathrm{Res}^m\colon H^*(S_0, \Sigma) \to H^*(S_1, \Sigma) \to \cdots \to H^*(S_{i-1}, \Sigma) \to$$
$$\to H^*(S_{i+1}, \Sigma) \to \cdots \to H^*(S_m, \Sigma) \tag{4.28}$$

defining the *compound class-residue.*

We have

THEOREM 20.6. *Let the cycle* γ *belong to the homology class* $h_c(S, \Sigma)$ *and have dimension* $D(\gamma) = l - 1$. *Then for the closed form* $\phi \in \mathcal{C}^\infty$ *on the manifold* S_0 *we have*

$$\int_{\delta\gamma \in h_c(S_0, \Sigma)} \phi = (2\pi i)^m \int_{\gamma \in h_c(S, \Sigma)} \operatorname{Res}^m [\phi]. \tag{4.29}$$

Equation (4.29) is obtained by successive applications of Theorem 20.5 to cycles belonging to the homology classes formed by the groups $H_c(S_i, \Sigma)$.

We now consider the case when the manifolds $S^{(i)}$ are given by equations of the form

$$(S^{(i)})\quad s_i(z) = 0,\ i = 1, \cdots, m,$$

where all functions $s_i(z)$ are holomorphic in some neighborhood $\widetilde{S}^{(i)}$ of the manifold $S^{(i)}$. Suppose that $\omega(z) \in \mathcal{C}^\infty$ for $z \in Z$ and satisfies the conditions

$$d\omega = 0,\ ds_i \wedge \omega = 0,\ i = 1, \cdots, m$$

(these conditions are analogous to the conditions (4.21_1) for $m = 1$).

Then it turns out to be possible, following I. M. Gel'fand and G. E. Šilov,[1] to construct in some neighborhood of the manifold S a system of forms

$$\widetilde{\omega}_{00\cdots 0}, \widetilde{\omega}_{10\cdots 0}, \widetilde{\omega}_{01\cdots 0}, \cdots, \widetilde{\omega}_{p_1\cdots p_m}, \cdots \in \mathcal{C}^\infty,$$

such that

$$\omega = ds_1 \wedge \cdots \wedge ds_m \wedge \widetilde{\omega}_{00\cdots 0},$$

$$d\widetilde{\omega}_{00\cdots 0} = ds_1 \wedge \widetilde{\omega}_{10\cdots 0} + ds_2 \wedge \widetilde{\omega}_{01\cdots 0} + \cdots + ds_m \wedge \widetilde{\omega}_{00\cdots 1},$$

$$d\widetilde{\omega}_{10\cdots 0} = ds_1 \wedge \widetilde{\omega}_{20\cdots 0} + ds_2 \wedge \widetilde{\omega}_{11\cdots 0} + \cdots + ds_m \wedge \widetilde{\omega}_{10\cdots 1}$$

and in general

$$d\widetilde{\omega}_{p_1\cdots p_m} = ds_1 \wedge \widetilde{\omega}_{p_1+1\cdots p_m} + \cdots + ds_m \wedge \widetilde{\omega}_{p_1\cdots p_m+1}.$$

These forms are defined in a non-unique way, but we have nevertheless the inclusion relations[2]

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1) See I. M. Gel'fand and G. E. Šilov, *Generalized functions and operations on them*, Vol. I, Fizmatgiz., Moscow, 1958 (Russian); German transl., *Verallgemeinerte Funktionen (Distributionen)*, I, VEB Deutscher Verlag der Wissenschaften, Berlin, 1960.

2) Further information about the theory of residues can be found in Leray [1]; see also Južakov [1].

$$\tilde{\omega}_{p_1 \cdots p_m}\Big|_S \in p_1! \cdots p_m! \operatorname{Res}\left[\frac{\omega}{s_1^{p_1+1} \cdots s_m^{p_m+1}}\right].$$

6. Generalization of Morera's theorem. Just as in the case of one variable, we have in the theory of several variables a theorem (known as Morera's theorem) which is the converse of the fundamental theorem of Cauchy-Poincaré. We cite this theorem, limiting ourselves to the simplest case, the space C^2 of the complex variables w, z.

THEOREM 20.7. *If the function* $f(w, z)$ *is continuous in a domain* $D \subset C^2$ *and the integral* $\int_S f(w, z)\, dw \wedge dz$ *vanishes when taken over the complete boundary of an arbitrary cylindrical domain* Q *of the form* $l_1 \times L_2$ *or* $l_2 \times L_1$ *and of class* A, *then the function* $f(w, z)$ *is holomorphic in the domain* D.

EXPLANATION. Here, if $Q = l_1 \times L_2$, then l_1 is a piecewise smooth line joining the points w_1 and w_2 in the w-plane, and L_2 is a simply-connected domain in the z-plane bounded by the piecewise smooth line l_2. If $Q = l_2 \times L_1$, then l_2 is a piecewise smooth line joining the points z_1 and z_2 in the z-plane, and L_1 is a simply-connected domain in the w-plane, bounded by the piecewise smooth contour l_1. The domains Q of this type make up the class A.

REMARK. As is evident from the formulation of the above theorem, we can derive a conclusion converse to the Cauchy-Poincaré theorem without requiring that the integral $\int\int_S f(w, z)\, dw \wedge dz$ vanish over all closed surfaces lying in the domain D.

PROOF. Consider an arbitrary point (w_0, z_0) of the domain D and the bicylinder $E = E_1 \times E_2 = \{|w - w_0| < r,\ |z - z_0| < r\}$ contained in D. It is clear that we have only to show that the function $f(w, z)$ is holomorphic in the bicylinder E.

Let the cylindrical domain Q and its boundary be contained within the bicylinder E. The boundary S of the cylindrical domain consists of: 1) the surface $l_1 \times l_2$; 2) the domain L_2 in the plane $w = w_1$; 3) the domain L_2 in the plane $w = w_2$. Since the integral of any continuous function over an analytic surface is always equal to zero,[1)] we find in integrating over S that the integrals over the

1) See subsection 7, § 4, Chapter I.

portions of S that lie in the planes $w = w_1$ and $w = w_2$ vanish, and we are left with

$$\int_{l_1 \times l_2}\int f(w, z)\, dw \wedge dz = \oint_{l_2} dz \int_{w_1\, l_1}^{w_2} f(w, z)\, dw = 0. \tag{4.30}$$

From this, and from the analogous equation resulting from the interchange of w and z, it follows that if λ_1 and λ_2 are two piecewise smooth lines lying respectively in the circles E_1 and E_2 and running from a to w and from b to z, then

$$F(w, z) = \int_{a\, \lambda_1}^{w} dw \int_{b\, \lambda_2}^{z} f(w, z)\, dz = \int_{\lambda_1 \times \lambda_2}\int f(w, z)\, dw \wedge dz \tag{4.31}$$

will be a single-valued function in the bicylinder E (i.e., will not depend on λ_1 and λ_2).

The integral $\int_{l_1} f(w, z)\, dw$ (the path l_1 is fixed) defines a continuous function of the variable z in the circle E_2. The integral of this function along an arbitrary closed contour $l_2 \subset E_2$ is equal to zero, by (4.30). It follows that the integral $\int_{b\,\lambda_2}^{z} dz \int_{a\,\lambda_1}^{w} f(w, z)\, dw$ defines a holomorphic function of the variable z in the circle E_2. Thus we have shown that $F(w, z)$ is a holomorphic function of the variable z in the circle E_2 for every fixed value of w in the circle E_1.

In exactly the same way we show that $F(w, z)$ is a holomorphic function of the variable w in the circle E_1 for every fixed value of z in the circle E_2. Therefore, $F(w, z)$ is a holomorphic function of the variables w, z in the bicylinder E. But, by (4.32)

$$f(w, z) = \frac{\partial^2 F}{\partial w \partial z}.$$

It follows that the function $f(w, z)$ is holomorphic in the bicylinder E.

§21. APPLICATION OF THE METHODS OF POTENTIAL THEORY TO THE STUDY OF HOLOMORPHIC FORMS. THE INTEGRAL FORMULA OF BOCHNER-MARTINELLI

1. Dirichlet's problem. In the classical theory of holomorphic functions of one complex variable Cauchy's integral formula is equivalent to Green's formula,

which solves Dirichlet's problem for harmonic functions. The reason, as is well known, is that the real part of a holomorphic function is harmonic, and conversely every harmonic function is representable as the real part of some holomorphic function. Moreover, every function continuous on the boundary of a plane domain D may be regarded as the boundary value of a function harmonic in D.

Matters stand differently in the general case. The real and imaginary parts of a holomorphic function of the complex variables $z_k = x_k + iy_k$, $k = 1, \cdots, n$, are pluriharmonic functions of the variables x_k, y_k, i.e., satisfy partial differential equations making up a system of the type of (1.32). An arbitrary function continuous on some closed surface $T = T_{2n-1}$ which forms the boundary of a closed domain D can always be considered as the value assumed on the boundary T by a function harmonic in the domain D. Pluriharmonic functions evidently constitute a subclass of the harmonic functions; therefore, generally speaking, there does not exist a pluriharmonic function in the closed domain $\bar{D}$ taking on pre-assigned values on the surface T.

Quite naturally then, the question arises as to whether it is possible to construct a pluriharmonic function in the closed domain $\bar{D}$ which shall take on values pre-assigned over some portion of the boundary of D. When the domain D is the bicylinder $\{|w| < 1, |z| < 1\}$ in the space of the variables $w = u + iv$, $z = x + iy$, and the part of the boundary in question is its skeleton $\Delta = \{w = e^{i\phi}, z = e^{i\psi}, 0 \leq \phi < 2\pi, 0 \leq \psi < 2\pi\}$, the answer to the question is in the negative. Given a function $\Phi(e^{i\phi}, e^{i\psi})$ defined on the surface Δ, we can construct a function $\Phi(w, z)$ which is continuous in the closed domain $\bar{D}$, satisfies throughout the domain D the equations

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0, \quad \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{4.32}$$

(such a function will be called double-harmonic), and assumes on the surface T values assigned in advance. This function may be obtained by a twofold application of Poisson's formula; it is then expressible as:

$$\Phi(w, z) = \Phi(re^{i\phi}, se^{i\psi}) =$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \Phi(e^{i\phi'}, e^{i\psi'}) P(r, \phi - \phi') P(s, \psi - \psi')\, d\phi' d\psi'. \tag{4.33}$$

Here

$$P(q, t) = \frac{1 - q^2}{1 + q^2 - 2q \cos t}.$$

In order that $\Phi(w, z)$ be a biharmonic function it must satisfy not only (4.32) but also the equations

$$\frac{\partial^2 \Phi}{\partial u \partial x} + \frac{\partial^2 \Phi}{\partial v \partial y} = 0; \quad \frac{\partial^2 \Phi}{\partial u \partial y} - \frac{\partial^2 \Phi}{\partial x \partial v} = 0. \tag{4.34}$$

For an arbitrary choice of the function $\Phi(e^{i\phi'}, e^{i\psi'})$ the function Φ defined by (4.33) will not in general satisfy (4.34) and so will not be biharmonic.

In order that a function defined by (4.33) be biharmonic, the values $\Phi(e^{i\phi'}, e^{i\psi'})$ must satisfy special conditions. By direct calculation we can show, for example, that a function defined by (4.33) will be biharmonic in the bicylinder D if for every $m, n = 1, 2, \cdots$ it satisfies the equations [1)]

$$\int_0^{2\pi} \int_0^{2\pi} \Phi(e^{i\phi'}, e^{i\psi'}) e^{i(m\phi' - n\psi')} \, d\phi' d\psi' = 0. \tag{4.35}$$

Thus in our case there is less connection than in the classical case between the properties of harmonic functions and of holomorphic functions of complex variables. The possibility of proving a given property for holomorphic functions by starting out with the theory of harmonic functions normally means that the property in question applies to a class of functions wider than the holomorphic functions we are studying.

When this actually occurs, the theory of harmonic functions is applicable and useful.

Such is the situation, for example, in the case of analytic extension of functions and of certain other questions in our theory, as studied by S. Bochner [1].

2. Analytic extension with the aid of Green's formula. For simplicity we consider the space of two complex variables w, z. We consider in this space certain p-dimensional (p may be 4, 3, or 2) simplexes $B_1, \cdots, B_s$ and the

1) The conditions for existence of a pluriharmonic function taking on pre-assigned values on the boundary of a domain D of more general form can be found in: Aĭzenberg [3], Rizza [1].

$(p-1)$-dimensional simplexes $C_1, \cdots, C_r$ which are their boundaries. All the C_m are assumed to be smooth. The simplexes B_q and C_m are supposed oriented in some definite fashion. We now define constants ϵ_{mq} in the following way.

If C_m is the boundary of B_q and the given orientation of C_m is coherent with the given orientation of B_q, then $\epsilon_{mq} = +1$; if the orientations are opposite, $\epsilon_{mq} = -1$. If C_m does not belong to the boundary of B_q, then $\epsilon_{mq} = 0$. We denote by B the set of all points belonging to the B_q, and by C the set of all points belonging to the C_m. Let there be given on the simplexes B_q continuous functions $\phi_q(u, v, x, y)$.

We shall say that these functions define in B a *composite* function $\phi(u, v, x, y)$ equal to $\phi_q(u, v, x, y)$ at the points of the simplex B_q. We denote this composite function by $\{\phi_q\}$. Later we consider on every simplex C_m the function

$$\phi^{\cdot}_m(u, v, x, y) = \sum_{q=1}^{s} \epsilon_{mq}\phi_q(u, v, x, y). \tag{4.36}$$

Such a function will be called the *saltus* of the function $\{\phi_q\}$ on C_m. It is clear that if C_m separates the simplexes B_a and B_b, then $\phi^{\cdot}_m = \pm(\phi_a - \phi_b)$: if C_m separates B_a from the exterior space, then $\phi^{\cdot}_m = \pm\phi_a$.

We take $p = 4$ and consider the exterior differential form

$$\begin{aligned}\alpha = U dv \wedge dx \wedge dy + V dx \wedge dy \wedge du + \\ + X dy \wedge du \wedge dv + Y du \wedge dv \wedge dx,\end{aligned} \tag{4.37}$$

where U, V, X, Y are functions of class $\mathcal{C}^1$ on all simplexes B_q. We apply Stokes' formula (4.4) to the differential of this form on each simplex B_q. We add the resulting equations and obtain the following relation:

$$\begin{aligned}\sum_{q=1}^{s} \int_{B_q} \left[\frac{\partial U_q}{\partial u} - \frac{\partial V_q}{\partial v} + \frac{\partial X_q}{\partial x} - \frac{\partial Y_q}{\partial y}\right] d\omega = \\ = \sum_{m=1}^{r} \int_{C_m} (U^{\cdot}_m dv \wedge dx \wedge dy + V^{\cdot}_m dx \wedge dy \wedge du + \\ + X^{\cdot}_m dy \wedge du \wedge dv + Y^{\cdot}_m du \wedge dv \wedge dx).\end{aligned} \tag{4.38}$$

Here $U^{\cdot}_m$, $V^{\cdot}_m$, $X^{\cdot}_m$, $Y^{\cdot}_m$ are the saltuses of the composite functions $\{U_q\}$, $\{V_q\}$,

$\{X_q\}$, $\{Y_q\}$ on C_m, and $d\omega = du \wedge dv \wedge dx \wedge dy$.

Analogous relations are easily obtained for the cases $p = 3$ and $p = 2$.

For certain later operations it is convenient to designate the points of our space in a way different from what we have done up to now. We set $u = x_1$, $v = x_2$, $x = x_3$, $y = x_4$. Let

$$\sigma^{\alpha} = (-1)^{\alpha-1}\, dx_1 \wedge \cdots \wedge [dx_\alpha] \wedge \cdots \wedge dx_4,$$

$$\sigma^{\alpha\beta} = (-1)^{\alpha+\beta-1}\, dx_1 \wedge \cdots \wedge [dx_\alpha] \wedge \cdots \wedge [dx_\beta] \wedge \cdots \wedge dx_4,$$

$$\sigma^{\alpha\beta} = -\sigma^{\beta\alpha}, \quad \sigma^{\alpha\alpha} = 0,$$

where $\alpha < \beta$. We apply Stokes' formula (4.4) to the simplexes B_q of dimension $p = 3$ and to the differential forms of the type $Z_{ik}dx_i \wedge dx_k$, $i, k = 1, 2, 3, 4$. It is assumed here that all the functions Z_{ik} belong to the class $\mathcal{C}^1$ on the simplexes B_q. Adding the resulting equations, we obtain a series of relations

$$\sum_{q=1}^{s} \int_{B_q} \left[\frac{\partial Z_{34}^{(q)}}{\partial x_2}\, dx_2 \wedge dx_3 \wedge dx_4 + \frac{\partial Z_{34}^{(q)}}{\partial x_1}\, dx_1 \wedge dx_3 \wedge dx_4 \right] =$$

$$= \sum_{p=1}^{r} \int_{C_p} Z_{34}^{\cdot(p)}\, dx_3 \wedge dx_4, \qquad (4.39)$$

. . . .

All these equations can be written (this can be verified immediately) in the single form (we denote the function under the integral sign by $\{U_q\}$):

$$\sum_{q=1}^{s} \int_{B_q} \left[\frac{\partial U_q}{\partial x_\beta}\, \sigma^{\alpha} - \frac{\partial U_q}{\partial x_\alpha}\, \sigma^{\beta} \right] = \sum_{m=1}^{r} \int_{C_m} U_p^{\cdot}\, \sigma^{\alpha\beta}. \qquad (4.40)$$

After these preliminary remarks we return to the basic exposition.

Let D be an ensemble of bounded domains (in particular it may consist of one domain) made up of a finite set of simplexes, with a piecewise smooth boundary $\partial D = B$. The simplexes that make up the domains of D are assumed positively oriented, and the orientation of the boundary B is coherent with the orientation of D; the function $f(x_1, x_2, x_3, x_4)$ is a real analytic function of its variables in some neighborhood $\tilde{D}$ of the set of closed domains $\bar{D}$, and

$$g = -\frac{1}{2}\left[\sum_{\alpha=1}^{4}(\xi_\alpha - x_\alpha)^2\right]^{-1} + H, \tag{4.41}$$

where H is a real analytic function of the variables x_α, ξ_α in the whole closed space. Then the expression

$$I = \int_B \sum_{\alpha=1}^{4}\left[f\frac{\partial g}{\partial \xi_\alpha} - g\frac{\partial f}{\partial \xi_\alpha}\right]\sigma^\alpha - \int_D (f\Delta g - g\Delta f)\,d\Omega =$$

$$= \int_B \sum_{\alpha=1}^{4} f\frac{\partial g}{\partial \xi_\alpha}\sigma^\alpha - \int_D \left[f\Delta g + \sum_{\alpha=1}^{4}\frac{\partial f}{\partial \xi_\alpha}\frac{\partial g}{\partial \xi_\alpha}\right] d\Omega \tag{4.42}$$

(here $\xi_1, \xi_2, \xi_3, \xi_4$ are variables of integration, and $d\Omega = d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \wedge d\xi_4$) has the value $2\pi^2 f(x_1, x_2, x_3, x_4)$ if the point $(x_1, x_2, x_3, x_4) \in D$ and zero if the point $(x_1, x_2, x_3, x_4) \notin \bar{D}$. In the particular case when $\Delta f = \Delta g = 0$, i.e., when f and g are harmonic functions $(H = 0)$ we have

$$I = \int_B \sum_{\alpha=1}^{4}\left[f\frac{\partial g}{\partial \xi_\alpha} - g\frac{\partial f}{\partial \xi_\alpha}\right]\sigma^\alpha. \tag{4.43}$$

Formulas (4.42) and (4.43) are generally known as *Green's formulas.* [1)]

We now assume that in our space we are given s three-dimensional finite simplexes B_q with r smooth faces C_m. Suppose that on the ensemble of simplexes $B = \partial D$ we are given a composite harmonic function $\{f_q\}$. We construct the integral

$$F(x_1, x_2, x_3, x_4) =$$

$$= \frac{1}{2\pi^2}\sum_{q=1}^{s}\int_{B_q}\sum_{\alpha=1}^{4}\left[f_q\frac{\partial g}{\partial \xi_\alpha} - g\frac{\partial f_q}{\partial \xi_\alpha}\right]\sigma^\alpha = Lf. \tag{4.44}$$

The complement $Rq\backslash\bar{B}$ consists of a finite set of domains $D_1, \cdots, D_t$; the domain D_∞ is infinite, and the rest are bounded. These bounded domains can in general be absent. The union $D_1 \cup \cdots \cup D_t \cup D_\infty$ of these domains will be

1) For a derivation of these formulas see, for example, Courant-Hilbert, *Methoden der Mathematischen Physik*, Vol. II, Interscience, New York, 1943, p. 237; English transl., Interscience, New York, 1962. The derivation is given for a three-dimensional space; an analogous result holds for our case.

denoted by D. The integral (4.44) defines a harmonic function $F_\tau(x_1, x_2, x_3, x_4)$ in each domain $D_\tau(\tau = 1, 2, \cdots, t, \infty)$; all F_τ, therefore, define in D a composite harmonic function $\{F_\tau\}$. The first terms of the integral (4.44) form a double-layer potential with density $-(1/2\pi^2) f_q$, and the second terms a single-layer potential. It is known from the general theory of harmonic functions,[1] that the saltus of such a two-layer potential on B_q will be f_q; a one-layer potential has no discontinuities as it traverses B_q. Therefore, the saltus of $\{F_\tau\}$ on B is $-\{f_q\}$.

We now prove a general theorem underlying our later derivations.

THEOREM 21.1. *Let $f(x_1, x_2, x_3, x_4)$ be a harmonic function in $\widetilde{B}$, and let the operator $\Omega\Phi$ be defined by the equation*

$$\Omega\Phi = \sum_\nu a_{\nu_1\nu_2\nu_3\nu_4} \frac{\partial^{\nu_1 + \cdots + \nu_4} \Phi}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \partial x_3^{\nu_3} \partial x_4^{\nu_4}}, \tag{4.45}$$

where $a_{\nu_1\nu_2\nu_3\nu_4}$ are constant coefficients. Then

$$L(\Omega f) = \Omega(Lf) \tag{4.46}$$

on condition that the saltus $\{f_q\}$ and all derivatives of $\{f_q\}$ up to order $n = \max(\nu_1 + \nu_2 + \nu_3 + \nu_4)$ inclusive vanish on all C_m. Here $Lf = F$ is defined by equation (4.44).

PROOF. It is evident that in order to obtain equation (4.46) we have only to show that

$$L\left[\frac{\partial f}{\partial \xi_\beta}\right] = \frac{\partial}{\partial x_\beta}(Lf), \tag{4.47}$$

and then iterate the operations of (4.47). We consider the difference $L(\partial f/\partial \xi_\beta) - (\partial/\partial x_\beta)(Lf)$. Taking into account the fact that $\partial g/\partial x_\beta = -\partial g/\partial \xi_\beta$, we obtain from (4.44) the equation

1) See Courant-Hilbert, *Methoden der Mathematischen Physik*, Vol. II, Interscience, New York, 1943, p. 238; English transl., Interscience, New York, 1962. The derivation there is for a three-dimensional space; an analogous fact holds in our case.

$$\Re_\beta = 2\pi^2 \left[L \left(\frac{\partial f}{\partial \xi_\beta} \right) - \frac{\partial}{\partial x_\beta} (Lf) \right] =$$

$$= \sum_{q=1}^{s} \int_{B_q} \sum_{\alpha=1}^{4} \left[\frac{\partial f_q}{\partial \xi_\beta} \frac{\partial g}{\partial \xi_\alpha} - g \frac{\partial^2 f}{\partial \xi_\alpha \partial \xi_\beta} + f \frac{\partial^2 g}{\partial \xi_\alpha \partial \xi_\beta} - \frac{\partial f_q}{\partial \xi_\alpha} \frac{\partial g}{\partial \xi_\beta} \right] \sigma^\alpha =$$

$$= \sum_{q=1}^{s} \int_{B_q} \sum_{\alpha=1}^{4} \frac{\partial}{\partial \xi_\beta} \left[f_q \frac{\partial g}{\partial \xi_\alpha} - g \frac{\partial f_q}{\partial \xi_\alpha} \right] \sigma^\alpha . \tag{4.48}$$

We further make use of the fact that f and g are harmonic functions, so that

$$\frac{\partial^2 f}{\partial \xi_\beta^2} = - \sum_{\alpha=1}^{4}{}' \frac{\partial^2 f}{\partial \xi_\alpha^2}, \quad \frac{\partial^2 g}{\partial \xi_\beta^2} = - \sum_{\alpha=1}^{4}{}' \frac{\partial^2 g}{\partial \xi_\alpha^2}$$

(here and later the symbol Σ' denotes that in the summation over α the value $\alpha = \beta$ is omitted). Then

$$\Re_\beta = \sum_{\alpha=1}^{4}{}' \left[\sum_{q=1}^{s} \int_{B_q} \frac{\partial}{\partial \xi_\beta} \left(f_q \frac{\partial g}{\partial \xi_\alpha} - g \frac{\partial f_q}{\partial \xi_\alpha} \right) \sigma^\alpha - \right.$$

$$\left. - \sum_{q=1}^{s} \int_{B_q} \frac{\partial}{\partial \xi_\alpha} \left(f_q \frac{\partial g}{\partial \xi_\alpha} - g \frac{\partial f_q}{\partial \xi_\alpha} \right) \sigma^\beta \right] =$$

$$= \sum_{p=1}^{r} \int_{C_p} \left(f_p^{\cdot} \frac{\partial g}{\partial \xi_\alpha} - g \left(\frac{\partial f}{\partial \xi_\alpha} \right)_p^{\cdot} \right) \sigma^{\alpha\beta} = 0. \tag{4.49}$$

The latter equation follows from formula (4.40). We obtain the zero from the assumption that the saltus $f_p^{\cdot}$, $(\partial f / \partial \xi_\alpha)_p^{\cdot}$ is equal to zero over every C_m. This completes the proof of the theorem.

COROLLARY. *If* 1) *f is a harmonic function in some neighborhood $\widetilde{B}$ of the set of simplexes B and satisfies there the equation $\Omega f = 0$, and* 2) *all saltuses of the function f and its partial derivatives (up to a suitable order) are zero on C, then the function $F = Lf$ is harmonic on the closed set D and satisfies $\Omega F = 0$ there.*

DEFINITION. We shall say that a differential operator $\Omega\Phi$ of the type (4.45) has the *uniqueness property* at infinity, if every analytic solution of the equation $\Omega\Phi = 0$ which is defined on the complement of some bounded domain

and vanishes at infinity is identically zero.

This property is possessed, for example, by the operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \quad (p < 4). \tag{4.50}$$

In fact, let the function $\Phi(x_1, \cdots, x_p, x_{p+1}, \cdots, x_4)$ be defined on the complement of the bounded domain E and let $\Delta\Phi = 0$. There always exists a system of values $x_{p+1}^0, \cdots, x_4^0$ and a $(4-p)$-dimensional neighborhood of these values such that for every system of values $x_{p+1}^*, \cdots, x_4^*$ in the selected neighborhood the function Φ is defined for $-\infty < x_k < \infty$ $(k = 1, \cdots, p)$. Therefore, the harmonic function of the first p variables $\Phi(x_1, \cdots, x_p, x_{p+1}^*, \cdots, x_4^*)$ is regular throughout the whole space and vanishes at infinity. Accordingly, it is equal to zero for every set of values $x_{p+1}^*, \cdots, x_4^*$ in the selected neighborhood and for arbitrary values of the first p variables. Since it is analytic in all four variables, it follows from the above that it is identically zero on the complement of the domain E.

Several consequences stem from this fact:

If a harmonic function f is defined in some neighborhood $\tilde{B}$ of a connected set of simplexes $B = \partial D$ (where D is a bounded domain), if f satisfies there the equation $\Omega f = 0$ (where the operator Ωf has the uniqueness property at infinity), and if the saltuses of the function f and the necessary partial derivatives are equal to zero on C_m, then the function $F = Lf$, harmonic in the domain D, takes on the same values on the boundary B as the function f.

It follows from the foregoing that in this case the function F is identically zero in the complement of the domain D taken in the whole space;[1)] further, the saltus of F on passing through the boundary B from D into the complement of D is equal to the value of f on the boundary, but with reversed sign. Our corollary follows from this.

Now if $f(w, z)$ is a holomorphic function of the variables w, z in some neighborhood $\tilde{B}$ of the boundary $\partial D = B$, then the real and imaginary parts of the function $f = U + iV$ satisfy the equations

---

1) That the function F vanishes at infinity follows immediately from the form of the function F as defined by formula (4.44).

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,\quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

(we return to our former notation for the variables) which, as we have seen, satisfy the condition of uniqueness at infinity. We may therefore apply the preceding corollary and obtain, using (4.44), functions harmonic in the domain D and coinciding in the neighborhood $\widetilde{B}$ with the functions U, V. But by the corollary, the functions so obtained are biharmonic in the domain D. By the uniqueness theorem of harmonic function theory, they are conjugate in the domain D. With their aid, we may obtain an analytic extension of the function f on the whole domain D. This proves the following theorem.

THEOREM 21.2. *Let D be a bounded domain with the connected boundary $\partial D = B$ in the space of the variables w, z. Every function $f(w, z)$ which is harmonic in some neighborhood $\widetilde{B}$ of the boundary B can be analytically extended to the whole domain.*

This theorem was first stated by Osgood in 1924.[1] Osgood's proof, however, was to some extent incomplete. The first complete proof was first published by Brown [1]. It is essentially different from the proof given here.

Theorem 21.2 (like the other results of this subsection) is valid for spaces of an arbitrary number of complex variables. It admits an extension to the case of meromorphic functions. The nature of the proof is in that case quite different, however.

3. **Generalization of Cauchy's integral formula.** We now consider the application of Green's formula to the development of a new generalization of Cauchy's integral formula. To this end we again look at formulas (4.42) and (4.43), writing them in terms of the complex variables w, z. In place of formula (4.42) we have

$$I = -\int_{\partial D}\left[f\frac{\partial g}{\partial \omega}\, d\omega\wedge d\bar{\zeta}\wedge d\zeta + g\,\frac{\partial f}{\partial\bar{\omega}}\, d\bar{\omega}\wedge d\bar{\zeta}\wedge d\zeta + \right.$$

$$\left. + f\frac{\partial g}{\partial \zeta}\, d\zeta\wedge d\bar{\omega}\wedge d\omega + g\,\frac{\partial f}{\partial\bar{\zeta}}\, d\bar{\zeta}\wedge d\bar{\omega}\wedge d\omega\right] +$$

$$+\frac{1}{16}\int_D (f\Delta g - g\Delta f)\, d\bar{\omega}\wedge d\omega\wedge d\bar{\zeta}\wedge d\zeta =$$

---

1) See Osgood [1], p. 206 ff.

$$= - \int_{\partial D} \left[f \frac{\partial g}{\partial \omega}\, d\omega \wedge d\bar{\zeta} \wedge d\zeta + g \frac{\partial f}{\partial \bar{\zeta}}\, d\zeta \wedge d\bar{\omega} \wedge d\omega \right] +$$

$$+ \frac{1}{16} \int_D \left[f \Delta g + 4 \left[\frac{\partial f}{\partial \bar{\omega}} \frac{\partial g}{\partial \omega} + \frac{\partial f}{\partial \bar{\zeta}} \frac{\partial g}{\partial \zeta} \right] \right] d\bar{\omega} \wedge d\omega \wedge d\bar{\zeta} \wedge d\zeta. \tag{4.51}$$

Here I, g have the same meaning as in formula (4.42) (only the functions f and H may take on complex values). In place of formula (4.43) we have

$$I = - \int_{\partial D} \left[f \frac{\partial g}{\partial \omega}\, d\omega \wedge d\bar{\zeta} \wedge d\zeta + g \frac{\partial f}{\partial \bar{\omega}}\, d\bar{\omega} \wedge d\bar{\zeta} \wedge d\zeta + \right.$$

$$\left. + f \frac{\partial g}{\partial \zeta}\, d\zeta \wedge d\bar{\omega} \wedge d\omega + g \frac{\partial f}{\partial \bar{\zeta}}\, d\bar{\zeta} \wedge d\bar{\omega} \wedge d\omega \right]. \tag{4.52}$$

Here again I has the same meaning as in formula (4.43); we must now write

$$\Delta f = 4 \left[\frac{\partial^2 f}{\partial w\, \partial \bar{w}} + \frac{\partial^2 f}{\partial z\, \partial \bar{z}} \right] = 0;$$

the function g and its derivatives are to be found from the equations

$$g = - \frac{1}{2} \left[|\omega - w|^2 + |\zeta - z|^2 \right]^{-1}; \quad \frac{\partial g}{\partial \omega} = 2 g^2 (\bar{\omega} - \bar{w}) \tag{4.53}$$

(here ω, ζ are variables of integration in the integrals (4.41) and (4.42)). In the particular case when the function $f(w, z)$ is holomorphic we have $\partial f / \partial \bar{w} = \partial f / \partial \bar{z} = 0$, and in place of (4.42) we obtain

$$I = - \int_{\partial D} \left[f \frac{\partial g}{\partial \omega}\, d\omega \wedge d\bar{\zeta} \wedge d\zeta + f \frac{\partial g}{\partial \zeta}\, d\zeta \wedge d\bar{\omega} \wedge d\omega \right]. \tag{4.54}$$

We thus arrive at the following theorem.

THEOREM 21.3 (Bochner-Martinelli).[1] *Let D be a bounded domain with a piecewise smooth boundary in the space C^2 of the complex variables w, z, and let $f(w, z)$ be a holomorphic function on the closed domain D. Then*

1) See Bochner [1], Martinelli [1].

$$\frac{1}{4\pi^2} \int_{\partial D} \frac{f(\omega, \zeta)[(\bar{w} - \bar{\omega})\, d\omega \wedge d\bar{\zeta} \wedge d\zeta + (\bar{z} - \bar{\zeta})\, d\zeta \wedge d\bar{\omega} \wedge d\omega]}{(|w - \omega|^2 + |z - \zeta|^2)^2} =$$

$$= \begin{cases} f(w, z) & \text{for } (w, z) \in D, \\ 0 & \text{for } (w, z) \notin \bar{D}. \end{cases} \tag{4.55}$$

Formula (4.55) expresses the value of $f(w, z)$ inside the domain D in terms of its value on the boundary of this domain. We may therefore think of it as a generalization of Cauchy's integral formula.

REMARK. The quantities $\bar{w}$, $\bar{z}$ entering the integral expression (4.55) must fall out during the calculation of the integral, since in the end the function $f(w, z)$ does not depend on them. With this in mind, we replace the quantities $\bar{w}$, $\bar{z}$ in (4.55) by w^*, z^*, where (w^*, z^*) is some point contained in a sufficiently small neighborhood of the point $(\bar{w}, \bar{z})$ (the point (w, z) is assumed to be an interior point of the domain D). Then the integral (4.55) defines for us some function $F(w, z, w^*, z^*)$ about which we know that the difference $F(w, z, w^*, z^*) - f(w, z)$ vanishes on the surface $w^* = \bar{w}$, $z^* = \bar{z}$. The latter equations define a non-analytic surface. The null surface of a holomorphic function which is not identically zero must be an analytic surface. It is therefore easy to conclude that the difference in question must vanish identically, and therefore $f(w, z) \equiv F(w, z, w^*, z^*)$. Here, however, the point (w^*, z^*) must lie in a sufficiently small neighborhood of the point $(\bar{w}, \bar{z})$, else the modified integral (4.55) may cease to exist.

Thus, for example, if the point $(0, 0) \in D$, then for some neighborhood of that point we set $w^* = z^* = 0$ and obtain

$$f(w, z) = -\frac{1}{4\pi^2} \int_{\partial D} f(\omega, \zeta) \frac{\bar{\omega} d\omega \wedge d\bar{\zeta} \wedge d\zeta + \bar{\zeta} d\zeta \wedge d\bar{\omega} \wedge d\omega}{[(w - \omega)\bar{\omega} + (z - \zeta)\bar{\zeta}]^2}. \tag{4.56}$$

The Bochner-Martinelli integral representation (4.55) is valid also in spaces of an arbitrary number of complex variables.

Let D be a bounded domain with a piecewise smooth boundary in the space C^n of the complex variables $z_1, \cdots, z_n$, and let $f(z) = f(z_1, \cdots, z_n)$ be a holomorphic function in the closed domain D. Then

$$\frac{(n-1)!}{(2\pi i)^n}\int_{\partial D}\frac{f(\zeta)\times\sum_{s=1}^{n}(\bar{\zeta}_s-\bar{z}_s)d\bar{\zeta}_1\wedge d\zeta_1\wedge\cdots\wedge[d\bar{\zeta}_s]\wedge d\zeta_s\wedge\cdots\wedge d\bar{\zeta}_n\wedge d\zeta_n}{(|\zeta_1-z_1|^2+\cdots+|\zeta_n-z_n|^2)^n}=$$

$$=\begin{cases} f(z) & \text{for } z\in D,\\ 0 & \text{for } z\in C^n\setminus\bar{D}.\end{cases} \tag{4.57}$$

For $n=1$ the Bochner-Martinelli formula (4.57) reduces to the classical integral formula of Cauchy. As in the case $n=2$, formula (4.57) for $n>2$ can be put in the form (4.56). Various properties of the integral representation (4.57) have been investigated by Look and Chung [1] and by V. A. Kakičev [2].

We now assume that $D=D_1\times\cdots\times D_q$, where D_ν is a bounded domain with a piecewise smooth boundary in the space of the variables $z_{k_{\nu-1}+1},\cdots,z_{k_\nu}$. Here $\nu=1,\cdots,q$, $k_0=0$, $k_q=n$. Then as a result of repeated application of formula (4.57) we obtain [1)]

$$f(z)=\frac{\prod_{\nu=1}^{q}(k_\nu-k_{\nu-1}-1)!}{(2\pi i)^n}\times$$

$$\times\int_{\partial D_1}\cdots\int_{\partial D_q} f(\zeta)\prod_{\nu=1}^{q}\frac{\sum_{\rho=k_{\nu-1}+1}^{k_\nu}(\bar{\zeta}_\rho-\bar{z}_\rho)d\bar{\zeta}_{k_{\nu-1}+1}\wedge d\zeta_{k_{\nu-1}+1}\cdots\wedge[d\bar{\zeta}_\rho]\wedge d\zeta_\rho\wedge\cdots\wedge d\bar{\zeta}_{k_\nu}\wedge d\zeta_{k_\nu}}{(|\zeta_{k_{\nu-1}+1}-z_{k_{\nu-1}+1}|^2+\cdots+|\zeta_{k_\nu}-z_{k_\nu}|^2)^{k_\nu-k_{\nu-1}}} \tag{4.58}$$

for all points $z\in D$. Here the integration is carried out on the product $\partial D_1\times\cdots\times\partial D_q$ of the boundaries of the domains $D_1,\cdots,D_q$. For $q=1$ formula (4.58) reduces to (4.57); for $q=n$ it reduces to Cauchy's integral formula (1.25) for the customary polycylindrical domains. The integral (4.58) vanishes at all points $z\in C^n\setminus\bar{D}$, except at those lying in the surfaces $\{z_{k_{\nu-1}+1}=z^0_{k_{\nu-1}+1},\cdots,z_{k_\nu}=z^0_{k_\nu}\}$, $(z^0_{k_{\nu-1}+1},\cdots,z^0_{k_\nu})\in\partial D_\nu$, $\nu=1,\cdots,q$. At points of these surfaces the integral (4.58) does not exist.

1) See Martinelli [1].

§22. THE BERGMAN-WEIL INTEGRAL FORMULA

Integral representations obtained by S. Bergman and A. Weil generalize Cauchy's integral formula (1.25) for polycylindrical domains to the case of the so-called analytic polyhedra.[1] These representations differ to some extent from each other; Bergman's formula is more general in character than Weil's. The latter, however, is more useful in applications; we will therefore derive Weil's integral representation below.[2]

1. *Analytic polyhedra.* Let D be a domain in the space C^n of the complex variables $z_1, \cdots, z_n$, and let $Z_1(z), \cdots, Z_N(z)$ be functions holomorphic in this domain. We consider in the plane of each of the complex variables Z_j $(j = 1, \cdots, N)$ a bounded domain D_j with boundary C_j, consisting of a finite set of arcs of class $\mathcal{C}^1$, and we form the closed sets $\Delta_j = \{z \in D,\ Z_j(z) \in D_j\}$. The portion of the boundary $\partial\Delta_j$ of the set Δ_j that lies in the domain D represents a certain set of hypersurfaces $\Gamma_j = \{z \in D,\ Z_j(z) \in C_j\}$.

The intersection of the closed sets Δ_j, $j = 1, \cdots, N$, consists of some set of domains. Let Δ be one of these, lying together with its boundary in the domain D: then its boundary $\partial\Delta = \sigma$ consists entirely of hypersurfaces belonging to the sets $\sigma_j = \sigma \cap \Gamma_j$; $j = 1, \cdots, N$; $\sigma = \bigcup_{j=1}^N \sigma_j$. If the intersections $\sigma_{j1} \cap \sigma_{j2}$ for all $j_1, j_2 = 1, \cdots, N$ are not themselves hypersurfaces, the domain Δ is said to be an *analytic polyhedron.*

In the sequel we shall suppose that all the domains D_j are required for the definition of the polyhedron Δ, i.e., for its definition we cannot use the intersection $\bigcap_{\substack{j=1 \\ j \neq j_0}}^N \Delta_j$ instead of the intersection $\bigcap_{j=1}^N \Delta_j$.

In subsection 4, §12 we introduced, for the space of two complex variables, the notion of analytic hypersurface, stratified into complex one-dimensional analytic surfaces. It is easy to see (for $n = 2$) that the hypersurfaces Γ_j have this property and are therefore analytic hypersurfaces.

Let Δ be a positively oriented analytic polyhedron. We assume the boundary $\partial\Delta = \sigma$ of this polyhedron, together with all its non-empty "faces" σ_j, to be coherently oriented. The boundary $\partial\sigma_j$ of a set of oriented hypersurfaces σ_j

1) See Bergman [1, 2], Weil [1, 2].

2) This derivation is due to F. Sommer [1].

consists in its turn of $(2n-2)$-dimensional "edges" $\sigma_{ji} = \partial\sigma_j \cap \partial\sigma_i$; $i = 1, \cdots, N$; $\partial\sigma_j = \bigcup_{i=1}^N \sigma_{ji}$. Here each non-empty set σ_{ji} is the union of topologically $(2n-2)$-dimensional surfaces. We assume that the boundary $\partial\sigma_j$ and all its non-empty parts σ_{ji} are oriented coherently with the orientation of σ_j. Then $\sigma_{ji} = -\sigma_{ij}$.

Next we consider the boundary $\partial\sigma_{ji}$ and its non-empty $(2n-3)$-dimensional edges $(\partial\sigma_{ji}) \cap \partial\sigma_s = \sigma_{jis}$, oriented coherently with σ_{ji}, and so on. The $(2n-k-1)$-*dimensional edges* of the polyhedron Δ are the sets $\sigma_{j1\cdots jkjk+1} = (\partial\sigma_{j_1\cdots j_k}) \cap (\partial\sigma_{j_{k+1}})$. Such a set is either empty or it represents an ensemble of topologically $(2n-k-1)$-dimensional surfaces oriented coherently with the orientation of $\sigma_{j_1\cdots j_k\cdots}$. It is obvious that

$$\partial\sigma_{j_1\cdots j_k} = \bigcup_{j_{k+1}=1}^{N} \sigma_{j_1\cdots j_k j_{k+1}} \tag{4.59}$$

and

$$\sigma_{j_1\cdots j_k} = \pm\, \sigma_{j_{\nu_1}\cdots j_{\nu_k}}, \tag{4.60}$$

where the plus sign corresponds to even, and the minus sign to odd, permutations $(\nu_1 \cdots \nu_k)$.

A particular example of an analytic polyhedron is afforded by the ordinary polycylindrical domain $\Delta = D_1 \times \cdots \times D_n \subset C_z^n$, where D_j is a bounded domain in the plane of the complex variable z_j with boundary C_j consisting of a finite set of real analytic arcs. Here $Z_j = z_j$; $j = 1, \cdots, n$; $N = n$. We assume that each domain D_j is positively oriented in its plane, and that the boundary C_j is coherently oriented. We write (as in §2, Chapter I) $S = C_1 \times \cdots \times C_n$ and postulate that the skeleton S of the domain Δ has the natural orientation of a product. We show that then

$$S = (-1)^{\frac{n(n-1)}{2}}\, \sigma_{12\cdots n}. \tag{4.61}$$

This relation can be obtained from general topological considerations. We shall give an elementary proof of it. By repeated application of Stokes' formula (4.4) we obtain:

$$\int_{\Delta^{(+)}} d\bar{z}_1 \wedge dz_1 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n = \sum_{j=1}^{n} \int_{\sigma_j} \bar{z}_1 dz_1 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n =$$

$$= \int_{\sigma_1} \bar{z}_1 dz_1 \wedge \overline{dz}_2 \wedge dz_2 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n =$$

$$= - \sum_{j=2}^{n} \int_{\sigma_{1j}} \bar{z}_1 \bar{z}_2 dz_1 \wedge dz_2 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n = \cdots$$

$$\cdots = (-1)^{\frac{n(n-1)}{2}} \int_{\sigma_{12\cdots n}} \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n dz_1 \wedge \cdots \wedge dz_n. \tag{4.62}$$

The first of these equations follows from the fact that $\partial\Delta^{(+)} = \Sigma_{j=1}^{n} \sigma_j$, $d(\bar{z}_1 dz_1 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n) = d\bar{z}_1 \wedge dz_1 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n$. The second equation holds because on C_j, and therefore on σ_j, we have $\overline{dz}_j \wedge dz_j = \overline{z_j'(t)}\, z_j'(t)\, \overline{dt} \wedge dt = 0$; here $z_j = z_j(t)$ is the equation of the line C_j.

It follows that in the second integral all terms except the first vanish. The third equation holds because on σ_{ij} we have $d(\bar{z}_1 \bar{z}_2 dz_1 \wedge dz_2 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n) = \bar{z}_1 \overline{dz}_2 \wedge dz_1 \wedge dz_2 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n = - z_1 dz_1 \wedge \overline{dz}_2 \wedge dz_2 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n$.

The remainder of the calculation is obvious.

On the other hand, we have

$$\int_{\Delta^{(+)}} \overline{dz}_1 \wedge dz_1 \wedge \cdots \wedge \overline{dz}_n \wedge dz_n =$$

$$= \int_{D_1^{(+)}} \overline{dz}_1 \wedge dz_1 \int_{D_2^{(+)}} \overline{dz}_2 \wedge dz_2 \cdots \int_{D_n^{(+)}} \overline{dz}_n \wedge dz_n =$$

$$= \int_{C_1} \bar{z}_1 dz_1 \int_{C_2} \bar{z}_2 dz_2 \cdots \int_{C_n} \bar{z}_n dz_n =$$

$$= \int_{S} \bar{z}_1 \cdots \bar{z}_n dz_1 \wedge \cdots \wedge dz_n. \tag{4.63}$$

Comparing equations (4.62) and (4.63), we obtain the relation (4.61).

We have the following theorem, which is important for the theory of analytic polyhedra:

THEOREM 22.1. *Let D_z be a domain of holomorphy in the space C_z^n of the variables $z_1, \cdots, z_n$, and let D_ζ be the same set in the space C_ζ^n of the complex variables $\zeta_1, \cdots, \zeta_n$. Then to every function $Z(z)$ holomorphic in D_z there correspond in the domain $D_\zeta \times D_z \subset C_\zeta^n \times C_z^n$ holomorphic functions*

$$P_1(\zeta, z), \cdots, P_n(\zeta, z),$$

such that

$$Z(\zeta) - Z(z) = \sum_{S=1}^{n} (\zeta_s - z_s) P_s(\zeta, z). \tag{4.64}$$

The proof of this theorem will be given in the second volume of this book. We remark that the theorem is obvious if the function $Z(z)$ is a polynomial (when instead of the domain $D_\zeta \times D_z$ we may choose an arbitrary finite domain in the space $C_\zeta^n \times C_z^n$). In this case we obtain the functions $P_s(\zeta, z)$ merely by expanding the difference $Z(\zeta) - Z(z)$ in a Taylor series. We collect all terms of this expansion, of whatever degree, containing the difference $\zeta_1 - z_1$, enclose the difference itself in brackets, and denote its coefficient by $P_1(\zeta, z)$. Then we select from the rest of the expansion the terms containing the difference $\zeta_2 - z_2$, enclose this difference in brackets and denote its coefficient by $P_2(\zeta, z)$. Continuing in this way, we arrive at the relation (4.64). In the general case Theorem 22.1 is not obvious and was proved by Hefer [1] in 1942.

We remark that the functions $P_s(\zeta, z)$ are not in general uniquely defined by specifying the function $Z(z)$; to every function $Z(z)$ there corresponds a set of representations (4.64). In the case just considered, when Z is a polynomial, this is obvious.

DEFINITION (skeleton of an analytic polyhedron). The set of topologically n-dimensional edges of the analytic polyhedron Δ

$$S = (-1)^{\frac{n(n-1)}{2}} \bigcup_{j_1 < \cdots < j_n} \sigma_{j_1 \cdots j_n} \tag{4.65}$$

is called its skeleton. The orientation of the skeleton is prescribed by equation (4.65).

The choice of this orientation is explained by the form of the integral representation of Weil (see Theorem 22.2 below).

We shall show in the second part of this book that the skeleton S of the

analytic polyhedron Δ contains its boundary in the sense of G. E. Šilov. This means that the modulus of any function holomorphic in the domain Δ and continuous on the closed domain $\bar{\Delta}$ attains its maximum value on the skeleton S. It was shown earlier (in subsection 2, §2, Chapter I) that the skeleton of an ordinary polycylindrical domain has this property.

2. **Weil's integral formula.**

THEOREM 22.2 (Weil [1, 2]). *Let* 1) Δ *be an analytic polyhedron defined by the holomorphic functions* $Z_j(z)$, $j = 1, \cdots, N$ (*where* $N \geq n$), *given in a holomorphy domain* D *of the space* C^n *of the variables* $z_1, \cdots, z_n$, *and by the domains* D_j, *bounded in the planes of the* Z_j *by the lines* C_j *belonging to the class* $\mathcal{C}^1$; *let* 2) $\sigma_{j_1, \cdots, j_n}$, *where* $j_1, \cdots, j_n = 1, \cdots, N$, *be the oriented, topologically n-dimensional, edges of this polyhedron; and let* 3) $f(z)$ *be a function holomorphic in the closed domain* $\bar{\Delta}$. *Then*

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_{j_1 < \cdots < j_n} \int_{\sigma_{j_1 \cdots j_n}} f(\zeta) D_{j_1 \cdots j_n} d\zeta =$$

$$= \begin{cases} f(z) \text{ for } z \in \Delta, \\ 0 \quad \text{for } z \in (D \setminus \bar{\Delta}) \setminus \bigcup_{j=1}^{N} \Gamma_j. \end{cases} \tag{4.66}$$

Here, as usual, $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$, $\Gamma_j = \{z \in D, Z_j(z) \in C_j\}$,

$$D_{j_1 \cdots j_n} = \begin{vmatrix} q_{j_1 1} \cdots q_{j_1 n} \\ \cdots\cdots\cdots \\ q_{j_n 1} \cdots q_{j_n n} \end{vmatrix}; \quad q_{ji}(\zeta, z) = \frac{P_{ji}(\zeta, z)}{Z_j(\zeta) - Z_j(z)} \tag{4.67}$$

($j = 1, \cdots, N$; $i = 1, \cdots, n$) *and the functions* $P_{ji}(\zeta, z)$ *are defined by the relations*

$$Z_j(\zeta) - Z_j(z) = \sum_{i=1}^{n} (\zeta_i - z_i) P_{ji}(\zeta, z), \; j = 1, \cdots, N, \tag{4.68}$$

in view of Theorem 22.1.

REMARK 1. For the polycylindrical domains considered in the preceding subsection we have $P_{ji} = 0$ for $j \neq i$ and $P_{jj} = 1$. The integral formula (4.66)

reduces to Cauchy's integral formula (1.25).

REMARK 2. In what follows we shall frequently make use of Weil's formula (4.66) for spaces of two complex variables $z_1 = w$, $z_2 = z$. In this case it has the form

$$\frac{1}{4\pi^2} \sum_{i<j} \int_{\sigma_{ij}} f(\omega, \zeta) \frac{(P_i Q_j - P_j Q_i)\, d\omega \wedge d\zeta}{(Z_i(\omega, \zeta) - Z_i(w, z))\,(Z_j(\omega, \zeta) - Z_j(w, z))} =$$

$$= \begin{cases} f(w, z) & \text{for } (w, z) \in \Delta, \\ 0 & \text{for } z \in (D \setminus \bar{\Delta}) \setminus \bigcup_{j=1}^{N} \Gamma_j. \end{cases} \qquad (4.66')$$

Here $P_i = P_{1i}(\omega, \zeta, w, z)$, $Q_i = P_{2i}(\omega, \zeta, w, z)$.

REMARK 3. It is obvious that in Theorem 22.2 the domain Δ can be replaced by a closed set consisting of a number of domains of the indicated type.

PROOF of Theorem 22.2. PART 1 (auxiliary). We assume that in the domain $G \subset C_z^n \times C_\zeta^n$ there are given functions $a_{\nu i}^{(j)}$, $b_{\nu i}^{(j)}$, c_{si}, where $i, \nu = 1, \cdots, n$; $j = (k+1), \cdots, n$; $s = 1, \cdots, k$. Here $k < n$. We use these functions to construct first $(n-k)n$ linear forms

$$r_i^{(j)} = \sum_{\nu=1}^{n} (a_{\nu i}^{(j)} d\zeta_\nu + b_{\nu i}^{(j)} \overline{d\zeta}_\nu), \qquad (4.69)$$

and then the form

$$D_{(n)}(c_1, \cdots, c_k, r^{(k+1)} \cdots r^{(n)}) = \begin{vmatrix} c_{11} \cdots c_{k1} & r_1^{(k+1)} \cdots r_1^{(n)} \\ c_{12} \cdots c_{k2} & r_2^{(k+1)} \cdots r_2^{(n)} \\ \cdots\cdots & \cdots\cdots \\ c_{1n} \cdots c_{kn} & r_n^{(k+1)} \cdots r_n^{(n)} \end{vmatrix}. \qquad (4.70)$$

The symbol on the left side of equation (4.70) indicates the ordering of the sequence of functions c_{is} and the forms $r_i^{(j)}$. The expression on the right is written as a determinant (the index for D prescribes the order). It is developed by the usual rule, but with the difference that the differentials are to be multiplied by exterior multiplication; when such a determinant is expanded into a sum the place of each factor in a term is determined by the number of the column in which it

lies. Therefore the determinant (4.70) vanishes if any row is a linear combination of the rest of the rows. This rule does not apply to columns, as for example in

$$D_n(d\zeta, \cdots, d\zeta) = \begin{vmatrix} d\zeta_1 & d\zeta_1 & \cdots & d\zeta_1 \\ d\zeta_2 & d\zeta_2 & \cdots & d\zeta_2 \\ \cdots & \cdots & \cdots & \cdots \\ d\zeta_n & d\zeta_n & \cdots & d\zeta_n \end{vmatrix} = n!\, d\zeta_1 \wedge \cdots \wedge d\zeta_n. \tag{4.71}$$

We also have the equations

$$D_{(n)}(c_1, \cdots, c_k, \cdots, r^{(l)}, \cdots) =$$
$$= D_{(n)}(c_1, \cdots, c_k, \cdots, r^{(l)} + \sum_{\nu=1}^{k} c_\nu \omega_\nu, \cdots), \tag{4.72}$$

$$D_{(n)}(c_1, \cdots, c_k, \cdots, gr^{(l)}, \cdots) =$$
$$= gD_{(n)}(c_1, \cdots, c_k, \cdots, r^{(l)}, \cdots). \tag{4.73}$$

Here g is a function given in the domain D, and $\omega_1, \omega_2, \cdots, \omega_k$ are linear differential forms of the type shown in (4.69).

We write further

$$\theta_i^{(k)} = r_i^{(k+1)} = \frac{1}{M^{n-k}}\, \overline{d\zeta_i} - \frac{(n-k)Q}{M^{n-k+1}}\, (\bar{\zeta}_i - \bar{z}_i), \tag{4.74}$$

where $k < n$, $i = 1, \cdots, n$.

$$M = \sum_{\nu=1}^{n} |\zeta_\nu - z_\nu|^2, \quad Q = \sum_{\nu=1}^{n} (\zeta_\nu - z_\nu)\, \overline{d\zeta_\nu}, \tag{4.75}$$

$$c_{si} = q_{j_s i}(\zeta, z), \tag{4.76}$$

where $s = 1, \cdots, n$, and $s_1, \cdots, s_n = 1, \cdots, N$ and

$$r_i^{(k+2)} = \cdots = r_i^{(n)} = d\zeta_i.$$

It is easy to see that

$$\begin{vmatrix} 1 & 1 \cdots 1 & -\dfrac{(n-k-1)Q}{M^{n-k}} & Q \cdots Q \\ \dfrac{\overline{\zeta}_1 - \overline{z}_1}{M} & c_{11} \cdots c_{k1} & \theta_1^{(k)} & \overline{d\zeta_1} \cdots \overline{d\zeta_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\overline{\zeta}_n - \overline{z}_n}{M} & c_{1n} \cdots c_{kn} & \theta_n^{(k)} & \overline{d\zeta_n} \cdots \overline{d\zeta_n} \end{vmatrix} = 0, \tag{4.77}$$

since the first row of the determinant is a linear combination of the remaining rows. To show this, we make use of the fact that

$$\sum_{i=1}^{n} c_{si}(\zeta_i - z_i) = \sum_{i=1}^{n} q_{j_s i}(\zeta_i - z_i) = 1.$$

Developing the determinant (4.77) by the elements of the first row we obtain

$$D_{(n)}(c_1, \cdots, c_k, \theta^{(k)}, \overline{d\zeta}, \cdots, \overline{d\zeta}) =$$

$$= \sum_{\rho=1}^{k} (-1)^{\rho-1} D_{(n)}\left[\frac{\overline{\zeta} - \overline{z}}{M}, c_1, \cdots, [c_\rho], \cdots, c_k, \theta^{(k)}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right] -$$

$$- (-1)^k \frac{n-k-1}{M^{n-k}} Q D_{(n)}\left[\frac{\overline{\zeta} - \overline{z}}{M}, c_1, \cdots, c_k, \overline{d\zeta}, \cdots, \overline{d\zeta}\right] +$$

$$+ (-1)^k (n-k-1) Q D_{(n)}\left[\frac{\overline{\zeta} - \overline{z}}{M}, c_1, \cdots, c_k, \theta^{(k)}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right]. \tag{4.78}$$

In this latter determinant we take out the factor $1/M^{n-k}$ from the $(k+2)$nd column; at the same time, we add to the elements of this column the elements of the first column multiplied by $(n-k)Q$. It is then evident that the last two terms of the expression (4.78) differ only as to sign, and the expression reduces to the first summation. Transforming each term of this sum as we did with the last determinant, we obtain the identity

$$D_{(n)}(c_1, \cdots, c_k, \theta^{(k)}, \overline{d\zeta}, \cdots, \overline{d\zeta}) =$$

$$= \sum_{\rho=1}^{k} (-1)^{k+\rho} D_{(n)}\left[c_1, \cdots, [c_\rho], \cdots, c_k, \frac{\overline{\zeta} - \overline{z}}{M^{n-k+1}}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right] \tag{4.79}$$

or (taking into account the substitution (4.76))

$$D_{(n)}(q_{j_1}, \cdots, q_{j_k}, \theta^{(k)}, \overline{d\zeta}, \cdots, \overline{d\zeta}) =$$

$$= \sum_{\rho=1}^{k} (-1)^{k+\rho} D_{(n)}\left[q_{j_1}, \cdots, [q_{j_\rho}], \cdots, q_{j_k}, \frac{\bar{\zeta} - \bar{z}}{M^{n-k+1}}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right]. \qquad (4.79')$$

PART 2. We return to the direct proof of formula (4.66). We begin with the integral formula of Bochner-Martinelli (4.57), assuming that the integration involved there is carried out over the boundary $\partial\Delta = \Sigma_{j=1}^{N} \sigma_j$. Using formula (4.71) and the notation introduced in Part 1 of the proof we obtain

$$\frac{(n-1)!}{M^n} \sum_{\rho=1}^{n} (\bar{\zeta}_\rho - \bar{z}_\rho) \overline{d\zeta_1} \wedge \cdots \wedge [\overline{d\zeta_\rho}] \wedge \cdots \wedge \overline{d\zeta_n} =$$

$$= D_{(n)}\left[\frac{\bar{\zeta} - \bar{z}}{M^n}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right]. \qquad (4.80)$$

With the aid of this relation we represent formula (4.57) in the form

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_j \int_{\sigma_j} f(\zeta) D_{(n)}\left[\frac{\bar{\zeta} - \bar{z}}{M^n}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right] \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n =$$

$$= \begin{cases} f(z) & \text{for } z \in \Delta, \\ 0 & \text{for } z \in C^n \backslash \bar{\Delta}. \end{cases} \qquad (4.81)$$

Theorem 22.2 will have been proved if we show that for all k satisfying the condition $1 \leq k \leq n$ we have the equation

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_{j_1 < \cdots < j_k} \int_{\sigma_{j_1 \cdots j_k}} f(\zeta) \sum_{\rho=1}^{k} (-1)^{k+\rho} D_{(n)}\left[q_{j_1}, \cdots, [q_{j_\rho}], \cdots\right.$$

$$\left. \cdots, q_{j_k}, \frac{\bar{\zeta} - \bar{z}}{M^{n-k+1}}, \overline{d\zeta}, \cdots, \overline{d\zeta}\right] \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n =$$

$$= \begin{cases} f(z) & \text{for } z \in \Delta, \\ 0 & \text{for } z \in E \backslash \bar{\Delta}, \end{cases} \qquad (4.82)$$

where $E = C^n$ for $k = 1$ and $E = D \setminus \bigcup_{j=1}^{N} \Gamma_j$ for $1 < k \le n$. The integration in formula (4.82) is carried out over the $(2n - k)$-dimensional edges (for the $(k-1)$-faces) of the polyhedron Δ.

For $k = 1$ equation (4.82) reduces to the Bochner-Martinelli formula. In this case the expression under the integral sign does not contain the quantities q_{js}, and therefore $E = C^n$; for $k > 1$ the integral (4.82) is defined only on

$$D \setminus \bigcup_{j=1}^{N} \Gamma_j.$$

For $k = n$ formula (4.82) coincides, in view of the identity (4.79′), with Weil's formula (4.66). Our theorem will be proved (by induction) if we show that formula (4.82) implies an analogous formula in which the integration over the $(2n - k)$-dimensional edges of the polyhedron Δ is replaced by integration over its $(2n - k - 1)$-dimensional edges.

Using the identity (4.79′), we represent the integral (4.82) in the form

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{k!\,(2\pi i)^n} \sum_{j_1, \cdots, j_k} \int_{\sigma_{j_1 \cdots j_k}} f(\zeta)\, D_{(n)}(q_{j_1}, \cdots, q_{j_k}, \theta^k, \overline{d\zeta}, \cdots$$
$$\cdots, \overline{d\zeta}) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n. \tag{4.83}$$

In obtaining (4.83) we have used the fact that

$$\sum_{j_1 < \cdots < j_k} = \frac{1}{k!} \sum_{j_1, \cdots, j_k = 1}^{N} . \tag{4.84}$$

Every $(2n - k - 1)$-dimensional surface $\sigma_{j_1 \cdots j_k j_{k+1}}$ is the boundary of $(k+1)$ surfaces $\sigma_{j_1 \cdots [j_\rho] \cdots j_{k+1}}$, $\rho = 1, \cdots, k+1$, and the bounding surface $(-1)^{k-\rho+1} \sigma_{j_1 \cdots j_{k+1}} = \sigma_{j_1 \cdots [j_\rho] \cdots j_{k+1} j_\rho}$ is oriented coherently with the surface $\sigma_{j_1 \cdots [j_\rho] \cdots j_{k+1}}$. Taking this into account, we represent the integral (4.83) in the form

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n\, k!\,(k+1)} \sum_{\rho=1}^{k+1} \sum_{j_1,\cdots,[j_\rho],\cdots,j_{k+1}} \int_{\sigma_{j_1\cdots[j_\rho]\cdots j_{k+1}}} f(\zeta)\, D(q_{j_1} \cdots$$

$$\cdots [q_{j_\rho}] \cdots q_{j_{k+1}}, \theta^{(k)}, \overline{d\zeta}, \cdots, \overline{d\zeta}) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n =$$

$$= \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n (k+1)!} \sum_{\rho=1}^{k+1} \sum_{j_1\cdots[j_\rho]\cdots j_{k+1}} \int_{\sigma_{j_1\cdots[j_\rho]\cdots j_{k+1}}} d\Big\{ f(z)\, D_{(n)}\Big[q_{j_1} \cdots$$

$$\cdots [q_{j_\rho}] \cdots q_{j_{k+1}}, \frac{\bar\zeta - \bar z}{M^{n-k}}, \overline{d\zeta}, \cdots, \overline{d\zeta}\Big] \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n \Big\}. \tag{4.85}$$

We apply Stokes' formula (4.4) to the integrals making up the latter sum. This is permissible, since on every surface $\sigma_{j_1\cdots[j_\rho]\cdots j_{k+1}}$ and on its boundary

$$\partial\sigma_{j_1\cdots[j_\rho]\cdots j_{k+1}} = \sum_{j_\rho=1}^{N} \sigma_{j_1\cdots[j_\rho]\cdots j_{k+1} j_\rho} =$$

$$= \sum_{j_\rho=1}^{N} (-1)^{k+1-\rho} \sigma_{j_1\cdots j_\rho \cdots j_{k+1}} \tag{4.86}$$

the expressions under the integral signs are of class $\mathcal{C}^1$ with respect to the variables ζ_i and $\bar\zeta_i$. As a result, we obtain instead of (4.82):

$$\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n (k+1)!} \sum_{\rho=1}^{k+1} \sum_{j_1\cdots[j_\rho]\cdots j_{k+1}} \sum_{j_\rho=1}^{N} \int_{\sigma_{j_1\cdots j_{k+1}}} f(\zeta)(-1)^{k+1-\rho} D\Big[q_{j_1} \cdots$$

$$\cdots [q_{j_\rho}] \cdots q_{j_{k+1}}, \frac{\bar\zeta - \bar z}{M^{n-k}}, \overline{d\zeta}, \cdots, \overline{d\zeta}\Big] \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n =$$

$$= \begin{cases} f(z) & \text{for } z \in \Delta, \\ 0 & \text{for } z \in E \setminus \overline{\Delta}. \end{cases} \tag{4.87}$$

In the expression (4.87) we now sum over the ordered indices $j_1, \cdots, j_{k+1}$ and put the summation over ρ under the integral sign. Then equation (4.87) is

transformed into formula (4.82) for the $(2n-k-1)$-dimensional edges of the analytic polyhedron Δ. With this, our Theorem 22.2 is proved.

REMARK 4. We note that formula (4.82) yields, in addition to the Bochner-Martinelli formula (for $k=1$) and the Weil formula (for $k=n$), a series of intermediate representations (for $1<k<n$). These also have applications in function theory.

REMARK 5. In formula (4.82) we may replace the variable $\bar{z}=(\bar{z}_1,\cdots,\bar{z}_n)$ by new variables $z'=(z'_1,\cdots,z'_n)$, independent of $z=(z_1,\cdots,z_n)$. In consequence, we obtain in place of (4.82) a new integral representation valid when the point z' lies in a sufficiently small neighborhood of the point $\bar{z}$.[1)]

3. **Weil's decomposition.** We consider an analytic polyhedron Δ of a special type: we assume that all domains D_i are disks $\{|z_j|<1\}$, $j=1,\cdots,N$. Such polyhedra we will call *Weil polyhedra* or *Weil domains*. For points z lying in some domain Δ_0, where $\overline{\Delta}_0\subset\Delta$ and the points $\zeta\in S$ (S is the skeleton of the polyhedron Δ), it is evident that we have

$$\frac{1}{[Z_{j_1}(\zeta)-Z_{j_1}(z)]\cdots[Z_{j_n}(\zeta)-Z_{j_n}(z)]}=$$

$$=\sum_{s_1,\cdots,s_n=0}^{\infty}\frac{[Z_{j_1}(z)]^{s_1}}{[Z_{j_1}(\zeta)]^{s_1+1}}\cdots\frac{[Z_{j_n}(z)]^{s_n}}{[Z_{j_n}(\zeta)]^{s_n+1}},\tag{4.88}$$

where $j_1,\cdots,j_n=1,\cdots,N$. This series converges uniformly if $z\in\Delta_0$ and $\zeta\in S$. Inserting this series into the expression (4.67) for $D_{j_1\ldots j_n}$, and then into Weil's formula (4.66) and integrating term-by-term, we find that for every function $f(z)$ holomorphic in the closed domain $\overline{\Delta}$ we have the equation

$$f(z)=\sum_{j_1<\cdots<j_n}\sum_{s_1,\cdots,s_n=0}^{\infty}P_{j_1\cdots j_n,\,s_1\cdots s_n}(Z_{j_1}(z))^{s_1}\cdots(Z_{j_n}(z))^{s_n}.\tag{4.89}$$

Here

1) See Sommer [1], Bochner [1].

$$P_{j_1\cdots j_n,\, s_1\cdots s_n}(z)=\frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n}\int_{\sigma_{j_1\cdots j_n}}\frac{\begin{vmatrix} p_{j_1 1}\cdots p_{j_1 n}\\ \cdots\cdots\cdots\\ p_{j_n 1}\cdots p_{j_n n}\end{vmatrix} f(\zeta)}{[Z_{j_1}(\zeta)]^{s_1+1}\cdots[Z_{j_n}(\zeta)]^{s_n+1}}\,d\zeta. \tag{4.90}$$

It is clear that these functions are holomorphic in the domain Δ.

As a particular case, if the $Z_j(z)$ are polynomials (when the domain Δ is usually called a Weil polynomial polyhedron) or are rational functions, then the p_{js} (and in consequence the $P_{j_1\cdots j_n,\, s_1\cdots s_n}$ also) are polynomials or rational functions. Then equation (4.89) yields a decomposition of an arbitrary function $f(z)$ holomorphic in the Weil domain Δ, into a series of polynomials or rational functions, and the series is uniformly convergent in the domain Δ. This proves the following two theorems, due to A. Weil.

THEOREM 22.3. *Every function $f(z)$ holomorphic in the Weil domain Δ can be represented in this domain by the series* (4.89). *This series converges uniformly in the domain Δ.*

THEOREM 22.4. *If the functions $Z_j(z)$, which by means of the inequalities $|Z_j|<1$, $j=1,\cdots,N$, define a Weil domain Δ, are either polynomials or rational functions, then the series* (4.89) *representing an arbitrary holomorphic function $f(z)$ in the domain Δ is a series of polynomials or of rational functions, respectively.*

§23. INTEGRAL REPRESENTATIONS IN DOMAINS OF SPECIAL TYPE

1. General integral representations in n-circular domains. [1]) Let D be a complete bounded n-circular domain in the space C^n of the complex variables $z_1,\cdots,z_n$ with center at the origin of coordinates, and let D^+ be its image in the absolute octant R_n^+ of the space R_n. We assume that the boundary ∂D of the domain D is a piecewise smooth hypersurface and we consider the stratification

$$\partial D=\bigcup_{|\zeta|\in\partial D^+}\Delta_{|\zeta|}.$$

1) See Aĭzenberg [4, 6].

Its base is the image ∂D^+ of the boundary ∂D, and the layers are the toruses $\Delta_{|\zeta|} = \{z_j = |\zeta_j|\, e^{i\theta_j},\ 0 \le \theta_j \le 2\pi,\ j = 1, \cdots, n,\ |\zeta| \in \partial D^+\}$. We define on the boundary ∂D^+ an odd continuous form of degree $n-1$, $\mu(|\zeta|)$ and a sequence of continuous functions $\{\psi_k(|\zeta|)\}$. Here k is a vector with the components $k_1, \cdots, k_n$ which take on the values $0, 1, 2, \cdots$.

It is easy to see that for a suitable orientation of the torus $\Delta_{|\zeta|}$ we have

$$\frac{1}{(2\pi i)^n} \int_{\Delta_{|\zeta|}} \zeta^k \bar{\zeta}^{\tilde{k}} \frac{d\zeta}{\zeta} = |\zeta|^k |\zeta|^{\tilde{k}} \delta_{k,\tilde{k}}.$$

Here, as usual, $\zeta^k = \zeta_1^{k_1} \cdots \zeta_n^{k_n}$, $\bar{\zeta}^{\tilde{k}} = \bar{\zeta}_1^{\tilde{k}_1} \cdots \bar{\zeta}_n^{\tilde{k}_n}$, $d\zeta/\zeta = d\zeta_1/\zeta_1 \wedge \cdots \wedge d\zeta_n/\zeta_n$,

$$\delta_{k,\tilde{k}} = \begin{cases} 0 & \text{for } k \ne \tilde{k}, \\ 1 & \text{for } k = \tilde{k}. \end{cases}$$

Supposing that the form $\mu(|\zeta|)$ and the functions $\psi_k(|\zeta|)$ are so chosen that

$$b_k = \left[\int_{\partial D^+} \psi_k(|\zeta|)\mu(|\zeta|)\right]^{-1} < \infty,$$

we obtain the following identity:

$$\frac{b_{\tilde{k}}}{(2\pi i)^n} \int_{\partial D^+} \frac{\psi_{\tilde{k}}(|\zeta|)}{|\zeta|^{2\tilde{k}}}\, \mu(|\zeta|) \int_{\Delta_{|\zeta|}} \zeta^k \bar{\zeta}^{\tilde{k}} \frac{d\zeta}{\zeta} = \delta_{k,\tilde{k}}.$$

We consider the function

$$H(z, \zeta) = \sum_k z^k \bar{\zeta}^k \frac{\psi_k(|\zeta|)}{|\zeta|^{2k}}\, b_k, \tag{4.91}$$

defined in the domain of convergence of the series (4.91). Then we have

THEOREM 23.1. *Suppose:*

1) *The series* (4.91) *converge for all* $z \in D$, $\zeta \in \partial D$.

2) *For every* $z \in D$ *the series* (4.91) *converge in the mean relative to repeated integration over the surfaces* $\Delta_{|\zeta|}$ *and* ∂D^+; *that is, for arbitrary* $\epsilon > 0$ *there exists an* $m_0 > 0$ *such that for* $m > m_0$

$$\int_{\partial D^+} |\mu(|\zeta|)| \int_{\Delta_{|\zeta|}} \left| H(z, \zeta) - \sum_{k_1, \cdots, k_n = 0}^{k_1, \cdots, k_n = m} z^k \bar{\zeta}^k \frac{\psi_k(|\zeta|)}{|\zeta|^{2k}} b_k \right| \frac{d|\zeta|}{|\zeta|} < \epsilon.$$

(Here $d|\zeta|/|\zeta| = d|\zeta_1|/|\zeta_1| \wedge \cdots \wedge d|\zeta_n|/|\zeta_n|$. The symbol $|\mu|$ means that after transformation under the integral sign to the coordinates on the surface ∂D^+ the function obtained from μ is to be replaced by its absolute value. Since μ is an odd form the choice of coordinate system on the surface ∂D^+ is not essential to the formulation of our condition.)

3) *The function* $f(z)$ *is holomorphic in the domain* D *and continuous on the closed domain* $\bar{D}$.

Then in the domain D *we have the integral representation*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D^+} \mu(|\zeta|) \int_{\Delta_{|\zeta|}} f(\zeta)\, H(z, \zeta) \frac{d\zeta}{\zeta}. \tag{4.92}$$

PROOF. I. We assume to begin with that the function $f(z)$ is holomorphic in the closed domain $\bar{D}$. Then this function can be represented in some domain $D_r \supset \bar{D}$ (where $r > 1$) by a power series (1.41) (see §3, Chapter I). Substituting this power series into the right-hand side of the equation to be verified (4.92), we easily convince ourselves of its validity.

II. We now assume that the function $f(z)$ is holomorphic in the domain D and continuous on the closed domain $\bar{D}$. Then the function $f(z)$ can be represented as the limit of a sequence $\{f_m(z),\ m = 1, 2, \cdots\}$ of functions holomorphic on the closed domain $\bar{D}$ and converging uniformly in this closed domain. For instance, we may take $f_m(z) = f(z(1 - (1/m + 1)))$. By Part I of the proof we have for every m

$$f_m(z) = \frac{1}{(2\pi i)^n} \int_{\partial D^+} \mu(|\zeta|) \int_{\Delta_{|\zeta|}} f_m(\zeta) H(z, \zeta) \frac{d\zeta}{\zeta}. \tag{4.93}$$

Passing to the limit with respect to m on both sides of equation (4.93) we obtain formula (4.92).

We remark that in virtue of its definition and of the first postulate of Theorem 23.1, the kernel $H(z, \zeta)$ of the integral representation (4.92) is a holomorphic function of the variables $z_1, \cdots, z_n$ in the domain D for $\zeta \in \partial D$.

We set $\psi_k(|\zeta|) = \phi^k(|\zeta|) = \phi_1^{k_1}(|\zeta|) \cdots \phi_n^{k_n}(|\zeta|)$, where $\phi_j(\zeta)$ are complex

continuous functions of the point $|\zeta|$, defined on the boundary ∂D^+.

Then the equation (4.91) has the form

$$H(z,\zeta)=\sum_k z^k\bar{\zeta}^k\frac{\phi^k(|\zeta|)}{|\zeta|^{2k}}b_k=$$

$$=h\left[z_1\bar{\zeta}_1\frac{\phi_1(|\zeta|)}{|\zeta_1|^2},\cdots,z_n\bar{\zeta}_n\frac{\phi_n(|\zeta|)}{|\zeta_n|^2}\right],\tag{4.91'}$$

and the integral representation (4.92) is replaced by

$$f(z)=\frac{1}{(2\pi i)^n}\int_{\partial D^+}\mu(|\zeta|)\times$$

$$\times\int_{\Delta_{|\zeta|}}f(\zeta)h\left[z_1\bar{\zeta}_1\frac{\phi_1(|\zeta|)}{|\zeta_1|^2},\cdots,z_n\bar{\zeta}_n\frac{\phi_n(|\zeta|)}{|\zeta_n|^2}\right]\frac{d\zeta}{\zeta}.\tag{4.92'}$$

THEOREM 23.2. *We consider a sequence of numbers b_k, where k is a vector with components $k_1,\cdots,k_n=0,1,2,\cdots$.*

If on the surface ∂D^+ there exists a continuous form $\mu(|\zeta|)$ such that the system of equations (this is the "moment problem")

$$\frac{1}{b_k}=\int_{\partial D^+}\phi^k(|\zeta|)\mu(|\zeta|)\tag{4.94}$$

is soluble in the class of functions $\phi_j(|\zeta|)$ satisfying the conditions

1) $\overline{\lim\limits_{\|k\|\to\infty}}\sqrt[\|k\|]{\dfrac{|\phi|^k(|\zeta|)}{|\zeta|^k}|b_k|\,d_k(D)}\le 1$ *for all* $|\zeta|\in\partial D^+$,

2) $\overline{\lim\limits_{\|k\|\to\infty}}\sqrt[\|k\|]{|b_k|\,d_k(D)\displaystyle\int_{\partial D^+}\frac{|\phi|^k(|\zeta|)}{|\zeta|^k}|\mu(|\zeta|)|}\le 1$,

where $\|k\|=k_1+\cdots+k_n$, $d_k(D)=\sup|z|^k$, then the series

$$\sum_k^{z\in D} b_k\mathfrak{z}^k=h(\mathfrak{z})\tag{4.95}$$

converges and defines a kernel of the integral representation (4.92′).

If this last occurs, condition 1) *is automatically satisfied.*

PROOF. In view of the corollary of Theorem 3.10, condition 1) of Theorem 23.2 is equivalent to condition 1) of Theorem 23.1. If condition 2) of Theorem 23.2 is satisfied, then so is condition 2) of Theorem 23.1.

If the function $h(z)$ can be used as a kernel in the integral representation (4.92′), then necessarily all $b_k \neq 0$. Therefore the system of equations (4.94) is meaningful. It is clear that condition 1) is necessary for the validity of our theorem.

We now suppose that for all $|\zeta| \in \partial D^+$ the functions $\phi_j(|\zeta|) \geq 0$ and the form $\mu(|\zeta|) \geq 0$. This last means that after transformation under the integral sign to the coordinates on the surface ∂D^+ the functions obtained from the form μ are non-negative. It is obvious that the choice of coordinate system has no influence, since μ is an odd form and we have supposed $\mu(|\zeta|) \geq 0$. Moreover, we require that the set of zeros of the form $\mu(|\zeta|)$ consist of isolated points. Then we have

THEOREM 23.3. *If on the surface ∂D^+ a non-negative continuous form $\mu(|\zeta|)$ is defined such that the system of equations (moment problem)* (4.94) *is soluble in the class of non-negative continuous functions $\phi_j(|\zeta|)$, satisfying for all vectors k the condition*

$$\frac{\max\limits_{|\zeta| \in \partial D^+} \phi^k(|\zeta|)}{\max\limits_{|\zeta| \in \partial D^+} |\zeta|^k} = \max_{|\zeta| \in \partial D^+} \frac{\phi^k(|\zeta|)}{|\zeta|^k}, \tag{4.96}$$

then the series (4.95) *converges and defines a kernel of the integral representation* (4.92′).

PROOF. For an arbitrary non-negative continuous function $F(|\zeta|)$, $|\zeta| \in \partial D^+$, we have the equation [1)]

$$\lim_{m \to \infty} \sqrt[m]{\int_{\partial D^+} F^m(|\zeta|)\, \mu(|\zeta|)} = \max F(|\zeta|). \tag{4.97}$$

1) See, for example, Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, England, 1934; Russian transl., IL, Moscow, 1948; p. 173. This equation is given there for the one-dimensional case; it is easily carried over to the general case.

Therefore the equation (4.96) is equivalent, when the conditions of Theorem 23.3 are satisfied, to conditions 1) and 2) of Theorem 23.2.

We now indicate a method of applying these theorems in practice. We choose an n-circular domain D_0 satisfying the conditions laid down at the beginning of this subsection. We set up on it the functions $\phi_j(|\zeta|) = |\zeta_j|^2$, $j = 1, \cdots, n$. Then, by Theorem 23.3, we have for this domain D_0 the integral representation

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_0^+} \mu_0(|\zeta|) \int_{\Delta_{|\zeta|}} f(\zeta)\, h_0(z_1\bar{\zeta}_1, \cdots, z_n\bar{\zeta}_n)\, \frac{d\zeta}{\zeta}. \tag{4.92''}$$

Here the kernel h_0 is defined by the convergent series

$$h_0(\mathfrak{z}) = \sum_k c_k \mathfrak{z}^k, \tag{4.98}$$

where $\mathfrak{z}$ is the point $(\mathfrak{z}_1\bar{\zeta}_1, \cdots, \mathfrak{z}_n\bar{\zeta}_n)$ and

$$\frac{1}{c_k} = \int_{\partial D_0^+} |\zeta|^{2k}\, \mu_0(|\zeta|). \tag{4.99}$$

It is a holomorphic function of the point z in the domain D_0 and of the point $\bar{\zeta}$ in the domain ∂D_0. We call the function $h_0(\mathfrak{z})$ the *Szegö kernel of the domain* D_0, by analogy with the one-dimensional case. In this method the domain D_0 is usually chosen to be simple enough so that the series (4.98) can be summed or that some other way can be found to obtain the kernel $h_0(\mathfrak{z})$ in closed form.

Then we seek an n-circular domain D for which, after the functions $\phi_j(|\zeta|)$ and the form $\mu(|\zeta|)$ have been suitably chosen, the integral representation (4.92′) is valid with $h(\mathfrak{z}) = h_0(\mathfrak{z})$.

The functions $\phi_j(|\zeta|)$ and the form $\mu(|\zeta|)$ may always be chosen so that the coefficients c_k in (4.99) are equal to the corresponding b_k in (4.94). Therefore, the representation in question will be valid in the domain D if, for the indicated choice of the functions $\phi_j(|\zeta|)$ and the form $\mu_0(|\zeta|)$, equation (4.96) is satisfied.

By this method one may obtain integral representations for an extensive class of n-circular domains.

2. **The integral formula of Temljakov.** We consider for simplicity the case of two variables z_1, z_2. We assume that the boundary of the domain D_0 has the form $\partial D_0\{|z_2|^2 = \Psi(|z_1|^2)\}$, and the boundary $\partial D\{|z_2| = \Phi(|z_1|)\}$. We suppose that the functions Φ and Ψ are twice continuously differentiable. We define on

the boundary ∂D_0^+ a non-negative form $\mu_0(|\zeta|)$, vanishing only at a set of isolated points. Then we have

THEOREM 23.4 (Aĭzenberg [6]). *Suppose:*

1) *The domain* D_0^+ *is strictly logarithmically convex, that is* $d^2 \ln \Psi(|z_1|^2)/(d \ln(|z_1|^2))^2 < 0$ *or else* $\Psi\Psi' + |z_1|^2 \Psi\Psi'' - |z_1|^2 \Psi'^2 < 0$.

2) *For the function* $\phi_1(|z_1|)$, *a solution of the functional equation*

$$\phi_1 \Psi'(\phi_1)\Phi(|z_1|) - |z_1| \Phi'(|z_1|)\Psi(\phi_1) = 0$$

(*the equation is solvable, because of condition* 1)), *the inequality*

$$\frac{d^2 \ln \Phi(|z_1|)}{(d \ln |z_1|)^2} < \frac{d^2 \ln \Psi(\phi_1)}{(d \ln \phi_1)^2}$$

is satisfied.

Then in the domain D *there exists an integral representation* (4.92′) *with the kernel* $h(\mathfrak{z}) = h_0(\mathfrak{z})$. *In this representation* $\phi_2 = \Psi(\phi_1)$ *and the form* $\mu(|\zeta|)$ *is obtained from the form* $\mu_0(|\zeta_0|)$ *by the substitution of variables* $|(\zeta_0)_1| = \phi_1(|\zeta_1|)$, $|(\zeta_0)_2| = \Psi(\phi_1(|\zeta_1|))$.

This theorem is an immediate consequence of Theorem 23.3. Condition 1) characterizes the domain D_0; condition 2) characterizes the domain D, on which we may use the method discussed above to extend the integral representation initially established on the domain D_0.

We now choose for the domain D_0 the hypersphere $\{|z_1|^2 + |z_2|^2 < 1\}$, and we set $\mu_0(|\zeta|) = 2|\zeta_1|\, d|\zeta_1|$. Then from formulas (4.98) and (4.99) we obtain

$$\frac{1}{c_k} = \int_0^1 |\zeta_1|^{2k_1}(1 - |\zeta_1|^2)^{k_2}\, 2|\zeta_1|\, d|\zeta_1| = \frac{k_1!\, k_2!}{(k_1 + k_2 + 1)!},$$

$$h_0(\mathfrak{z}) = \sum_k c_k \mathfrak{z}^k =$$

$$= \sum_{k_1, k_2 = 0}^{\infty} \frac{(k_1 + k_2 + 1)!}{k_1!\, k_2!} \mathfrak{z}_1^{k_1} \mathfrak{z}_2^{k_2} = \frac{1}{(1 - \mathfrak{z}_1 - \mathfrak{z}_2)^2}. \tag{4.100}$$

Thus for the hypersphere D_0 we have the integral representation

$$f(z) = \frac{2}{(2\pi i)^2} \int_0^1 |\zeta_1|\, d|\zeta_1| \int_{\Delta_{|\zeta|}} \frac{f(\zeta)}{(1 - z_1\bar{\zeta}_1 - z_2\bar{\zeta}_2)^2} \frac{d\zeta}{\zeta}. \tag{4.101}$$

On D we choose the convex domain $\{|z_2| < \Phi(|z_1|),\ 0 \le |z_1| \le r\}$, where the function Φ is supposed twice continuously differentiable. We write

$$\begin{gathered} \phi_1(|\zeta_1|) = \frac{|\zeta_1|\,\Phi'(|\zeta_1|)}{|\zeta_1|\,\Phi'(|\zeta_1|) - \Phi(|\zeta_1|)}, \\ \phi_2(|\zeta_2|) = 1 - \phi_1(|\zeta_1|),\ \ \mu(|\zeta_1|) = d\phi_1(|\zeta_1|). \end{gathered} \tag{4.102}$$

It is possible to show that the domains D_0 and D satisfy all conditions of Theorem 23.4. Therefore we have the following integral representation for the domain D:

$$f(z) = \frac{1}{(2\pi i)^2} \int_0^r \phi_1'(|\zeta_1|)\, d|\zeta_1| \times$$

$$\times \int_{\Delta_{|\zeta|}} \frac{f(\zeta)}{\left[1 - z_1\bar{\zeta}_1 \dfrac{\phi_1(|\zeta_1|)}{|\zeta_1|^2} - z_2\bar{\zeta}_2 \dfrac{\phi_2(|\zeta_2|)}{|\zeta_2|^2}\right]^2} \frac{d\zeta}{\zeta}, \tag{4.103}$$

where the functions ϕ_1 and ϕ_2 are defined by (4.102) and $|\zeta_2| = \Phi(|\zeta_1|)$.

What we have obtained is the integral formula of A. A. Temljakov. In order to bring it into the form in which Temljakov first stated it, we set $\tau = \phi_1(|\zeta_1|)$, $\eta = \zeta_1/|\zeta_1|$, $t = \arg\zeta_1 - \arg\zeta_2$. Then we have, in place of (4.103) [1)]

$$f(z) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\eta|=1} \frac{\eta f(r_1(\tau)\eta,\ r_2(\tau)\eta e^{-it})}{(\eta - u)^2}\, d\eta, \tag{4.104}$$

where we have set $|\zeta_1| = r_1(\tau)$,

$$u = \frac{\tau}{r_1(\tau)} z_1 + \frac{1-\tau}{r_2(\tau)} z_2 e^{it},\ \ |\zeta_2| = r_2(\tau) = \exp\left[-\int_0^\tau \frac{\tau}{1-\tau}\, d\ln r_1(\tau)\right]. \tag{4.105$_1$}$$

1) See Temljakov [2,3,4]. In these papers formulas (4.104) and (4.107) were obtained by a different method, and considerably earlier than Theorems 23.1–23.4.

Condition 2) of Theorem 23.4 will be satisfied if $r_1(0) = 0$, $r_1(1) < 1$, and

$$0 < r_1'(\tau) \leq \frac{r_1(\tau)}{\tau} \quad \text{for } 0 < \tau < 1. \tag{4.105$_2$}$$

We remark that the geometric meaning of condition (4.105) is as follows: In the absolute quadrant R_2^+ of the real variables v_1, v_2 the curve defined by the equations $v_k = r_k(\tau)$, $k = 1, 2$ $(0 \leq \tau \leq 1)$, is the envelope of the family of lines

$$\tau \frac{v_1}{r_1(\tau)} + (1 - \tau) \frac{v_2}{r_2(\tau)} = 1, \quad 0 \leq \tau \leq 1 \tag{4.106}$$

and lies under them.

We consider the operator $L^{(1,2)}[f(z)] = f + z_1 f'_{z_1} + z_2 f'_{z_2} = F(z)$. If a_k are the Taylor coefficients of the function $f(z)$, and b_k are those of the function $F(z)$, then $(k_1 + k_2 + 1)\, a_k = b_k$. Noting that

$$L^{(1,2)}\left[\frac{1}{\eta - u}\right] = \frac{\eta}{(\eta - u)^2},$$

and applying the operator inverse to $L^{(1,2)}$ to both sides of equation (4.104) (the effect of such an operator on a holomorphic function amounts to dividing its Taylor coefficients b_k by $(k_1 + k_2 + 1)$), we obtain

$$\Phi(z) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\eta|=1} \frac{f(r_1(\tau)\eta,\, r_2(\tau)\eta e^{-it})}{\eta - u}\, d\eta, \tag{4.107}$$

where $f = L^{(1,2)}[\Phi]$.

The integrals (4.107) and (4.104) will be referred to in the sequel as *Temljakov integrals of the first and second kind*, respectively. We extend these integrals to the case of n variables.

We consider the class (T) of n-circular bounded domains in the space C^n of the variables $z_1, \cdots, z_n$ with center at the origin of coordinates. We shall say that an n-circular domain $D \in (T)$ if it is bounded by the hypersurface

$$z_1 = r_1(\tau_1, \cdots, \tau_{n-1})\eta, \quad z_{k+1} = r_{k+1}(\tau_1, \cdots, \tau_{n-1})\eta e^{-it_k}, \tag{4.108}$$

where $|\eta| = 1$, $0 \leq \tau_k \leq 1$, $0 \leq t_k \leq 2\pi$, $k = 1, \cdots, n-1$ and the functions $r_1(\tau), \cdots, r_n(\tau)$ satisfy the conditions:

a) they are non-negative and continuous in all their variables;

b) in the absolute octant R_n^+ of the space R_n of the real variables $v_1, \cdots, v_n$ the topologically $(n-1)$-dimensional surfaces

$$v_s = r_s(\tau_1, \cdots, \tau_{n-1}), \quad s = 1, \cdots, n, \tag{4.109}$$

envelop the family of hypersurfaces

$$\sum_{s=1}^{n} \frac{\phi_s(\tau_1, \cdots, \tau_{n-1})}{r_s(\tau_1, \cdots, \tau_{n-1})} v_s = 1. \tag{4.110}$$

Here $0 \le \tau_k \le 1$, $k = 1, \cdots, n-1$,

$$\phi_s(\tau_1, \cdots, \tau_{n-1}) = (1 - \tau_{s-1}) \tau_s \tau_{s+1} \cdots \tau_n,$$

where $s = 1, \cdots, n$ and $\tau_0 = 0$;

c) in the absolute octant R_n^+ the hyperplanes (4.110) lie outside the domain abutting on the origin and bounded by the coordinate hyperplanes and the hypersurface (4.109).

We note that an n-circular domain $D \in (T)$ is always complete. We have [1)]

THEOREM 23.5. 1) *If the function $f(z)$ is holomorphic in the domain $D \in (T)$ and is continuous on the closed domain $\bar{D}$, then at points $z \in D$ we have a Temljakov integral representation of the second kind*

$$f(z) = \frac{1}{(2\pi)^n i} \int_0^{2\pi} dt_1 \cdots \int_0^{2\pi} dt_{n-1} \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_{n-1} \times$$

$$\times \int_{|\eta|=1} f(r_1(\tau)\eta, r_2(\tau)\eta e^{-it_1}, \cdots, r_n(\tau)\eta e^{-it_{n-1}}) L\left[\frac{1}{\eta - u}\right] d\eta. \tag{4.111}$$

Here

$$u = \sum_{s=1}^{n} \frac{\phi_s(\tau)}{r_s(\tau)} z_s e^{-it_{s-1}} \quad (\textit{where } t_0 \equiv 0),$$

and the operator $L(\Phi)$ is defined by the equation

$$L(\Phi) = L^{(1,2)} L^{(1,2,3)} \cdots L^{(1,2,\cdots,n)}(\Phi), \tag{4.112}$$

---

1) See Temljakov [2–4], Aĭzenberg [2]. The general case of n variables was considered in the latter paper.

where $L^{(1,2,\cdots,k)}(\Phi) = \Phi + z_1\Phi'_{z_1} + \cdots + z_k\Phi'_{z_k}$, $k \leq n$.

2) *If the function* $f(z)$ *is holomorphic in the domain* $D \in (T)$, *and the function* $F(z) = L(f)$, *where* L *is the operator* (4.112), *is holomorphic in the domain* D *and continuous on the closed domain* $\overline{D}$, *then for points* $z \in D$ *we have a Temljakov integral representation of the first kind*

$$f(z) = \frac{1}{(2\pi)^n i} \int_0^{2\pi} dt_1 \cdots \int_0^{2\pi} dt_{n-1} \int_0^{2\pi} d\tau_1 \cdots$$

$$\cdots \int_0^{2\pi} d\tau_{n-1} \int_{|\eta|=1} \frac{F(r_1(\tau)\eta, r_2(\tau)\eta e^{-it_1}, \ldots, r_n(\tau)\eta e^{-it_{n-1}})}{\eta - u} d\eta. \quad (4.113)$$

We remark that Theorem 23.5, as well as other generalizations of the Temljakov integrals to the case of arbitrarily many complex variables, 1) can be obtained by starting from the results of the preceding subsection of this section.

We say that the function $\Phi(z)$, holomorphic in the domain $D \in (T)$, belongs to the class $h_1(D)$ if

$$\lim_{\rho \to 1} \int_0^{2\pi} dt_1 \cdots \int_0^{2\pi} dt_{n-1} \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_{n-1} \cdots$$

$$\cdots \int_{|\eta|=1} |\Phi(r_1(\tau)\rho\eta, r_2(\tau)\rho\eta e^{-it_1}, \ldots, r_n(\tau)\rho\eta e^{-it_{n-1}})|\, |d\eta| < \infty. \quad (4.114)$$

L. A. Aĭzenberg [2] showed that formula (4.111) holds if $f(z) \in h_1(D)$, and formula (4.113) holds if $F(z) \in h_1(D)$.

The integrals (4.113) and (4.111), in which the functions F or f are replaced by an arbitrary summable function F defined on the boundary ∂D of a domain $D \in (T)$, will be referred to as integrals of Temljakov type of the first and second kinds, respectively. We limit our considerations to integrals of Temljakov type for the space C^2 of the variables z_1, z_2 and will construct them after the model of the integrals (4.104) and (4.107). We have

THEOREM 23.6 (Aĭzenberg [1]). *Suppose that on the boundary* ∂D *of the domain* $D \in (T)$, *a bounded hypersurface* (4.108) *with* $n = 2$, *there is given a summable (in the sense of Lebesgue) function* $F(\tau, t, \eta)$; $a = \lim_{\tau \to 0}(r'_2(\tau)/r'_1(\tau))$,

1) See Aĭzenberg [2], Li Če Gon [1].

$b = \lim_{\tau \to 1}(r_2'(\tau)/r_1'(\tau))$. *Then the following assertions are true:*

I. *If* $a \neq 0$, $b \neq -\infty$, *then the Temljakov type integral of the first kind*

$$\psi(z_1, z_2) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 dr \int_{|\eta|=1} \frac{F(t, r, \eta)}{\eta - u}\, d\eta \qquad (4.115)$$

and the Temljakov type integral of the second kind

$$\phi(z_1, z_2) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 dr \int_{|\eta|=1} \frac{\eta F(t, r, \eta)}{(\eta - u)^2}\, d\eta \qquad (4.116)$$

are holomorphic in the domains D, $E_1 = \{a|z_1| + |z_2| + r_2(0) < 0\}$, $E_2 = \{b|z_1| + |z_2| + br_1(1) > 0\}$ *and are not holomorphic in the domain* $C^2 \setminus \overline{(D \cup E_1 \cup E_2)}$.

II. *If* $a \neq 0$, $b = -\infty$, *then the integrals* (4.115) *and* (4.116) *are holomorphic in the domains* D *and* E_1 *and are not holomorphic in the domain* $C^2 \setminus \overline{(D \cup E_1)}$.

III. *If* $a = 0$, $b \neq -\infty$, *the integrals* (4.115) *and* (4.116) *are holomorphic in the domains* D *and* E_2 *and are not holomorphic in the domain* $C^2 \setminus \overline{(D \cup E_2)}$.

IV. *If* $a = 0$, $b = -\infty$, *the integrals* (4.115) *and* (4.116) *are holomorphic in the domain* D *and are not holomorphic in the domain* $C^2 \setminus \bar{D}$.

This theorem, in particular, characterizes the behavior of functions defined by Temljakov integrals outside the domain $D \in (T)$. The result we have obtained shows the essential difference between Temljakov's integrals and Cauchy's integral and its generalizations to the case of holomorphic functions of several variables (the Bochner-Martinelli and Bergman-Weil integrals).

We remark that in the integral representation (4.107) it is easy to go from the Cauchy kernel $(\eta - u)^{-1}$ to the Poisson kernel [1] of the form $(1 - \rho^2)/(1 + \rho^2 - 2\rho \cos(\theta - \phi))$, by setting $u = \rho e^{i\phi}$. By this method we may obtain the following result.

THEOREM 23.7. *If the real-valued function* $f(z_1, z_2)$ *is biharmonic in the the domain* $D \in (T)$, *and the function* $F(z_1, z_2) = f + x_1 f'_{x_1} + y_1 f'_{y_1} + x_2 f'_{x_2} + y_2 f'_{y_2}$, *where* $z_k = x_k + iy_k$, $k = 1, 2$, *is continuous in the closed domain* $\bar{D}$, *then*

$$f(z_1, z_2) = \frac{1}{4\pi^2} \int_0^{2\pi} dt \int_0^1 dr \int_0^{2\pi} \frac{(1 - \rho^2) F(r_1(r) e^{i\theta}, r_2(r) e^{i(\theta - t)})}{1 + \rho^2 - 2\rho \cos(\theta - \phi)}\, d\theta. \qquad (4.117)$$

---

1) See Aĭzenberg [2].

Here

$$u = \rho e^{i\phi} = \tau \frac{z_1}{r_1(\tau)} + (1-\tau)\frac{z_2 e^{it}}{r_2(\tau)}.$$

By a similar method one obtains an integral representation for biharmonic functions corresponding to the Temljakov integral formula of the second kind.

Similar statements hold also for pluriharmonic functions in n-circular domains of the class (T) in the space C^n. These integral representations turn out to be useful for obtaining the conditions of solvability for Dirichlet's problem in the class of pluriharmonic functions and for solving a number of other problems.[1)]

3. **Integral representations for certain classes of n-circular domains.** First of all we consider, for the two-variable case, certain other integral representations obtained by the method outlined at the end of subsection 1 and at the beginning of subsection 2 of this section.

a) We choose $D_0^{(p)} = \{|z_1|^{2/p} + |z_2|^2 < 1\}$, where p is a positive integer. We set $\mu_0(|\zeta|) = 2|\zeta_1|\, d\zeta_1$. Then from (4.98) and (4.99) we obtain

$$\frac{1}{c_k} = \int_0^1 |\zeta_1|^{2k_1}(1 - |\zeta_1|^{\frac{2}{p}})^{k_2}\, 2|\zeta_1|\, d|\zeta_1| = \frac{p[p(k_1+1)-1]!\,k_2!}{[p(k_1+1)+k_2]},$$

$$h_0(\mathfrak{z}) = \sum_k c_k \mathfrak{z}^k = \frac{(1-\mathfrak{z}_2)^{p-1}}{[(1-\mathfrak{z}_2)^p - \mathfrak{z}_1]^2}.$$

Thus, for the domain $D_0^{(p)}$ we obtain the integral representation

$$f(z) = \frac{2}{(2\pi i)^2}\int_0^1 |\zeta_1|\, d(|\zeta_1|) \int_{\Delta_{|\zeta|}} \frac{f(\zeta)(1 - z_2\bar{\zeta}_2)^{p-1}}{[(1 - z_2\bar{\zeta}_2)^p - z_1\bar{\zeta}_1]^2}\frac{d\zeta}{\zeta}.$$

If the domain $D^{(p)}$ $\{|z_2| < \Phi(z_1)\}$ satisfies the conditions that the function Φ is twice continuously differentiable and that $p|z_1|\,\Phi'' + (p-1)\Phi' < 0$, then by Theorem 23.4 we have the following integral representation in the domain D:

1) See Aĭzenberg [3].

$$f(z)=\frac{1}{(2\pi i)^2}\int_0^1 \phi_1'(|\zeta_1|)\,d|\zeta_1|\int_{\Delta_{|\zeta|}}\frac{f(\zeta)\left[1-z_2\bar{\zeta}_2\,\dfrac{\phi_2(|\zeta_2|)}{|\zeta_2|^2}\right]^{p-1}}{\left\{\left[1-z_2\bar{\zeta}_2\,\dfrac{\phi_2(|\zeta_2|)}{|\zeta_2|^2}\right]^p-z_1\bar{\zeta}_1\,\dfrac{\phi_1(|\zeta_1|)}{|\zeta_1|^2}\right\}^2}\frac{d\zeta}{\zeta},$$

where

$$\phi_1(|\zeta_1|)=\left[\frac{p|\zeta_1|\Phi'(|\zeta_1|)}{p|\zeta_1|\Phi'(|\zeta_1|)-\Phi(\zeta)}\right]^p,$$

$$\phi_2(|\zeta_2|)=\frac{-\Phi(|\zeta_1|)}{p|\zeta_1|\Phi'(|\zeta_1|)-\Phi(|\zeta_1|)}.$$

For $p=1$ this integral representation reduces to the Temljakov integral representation (4.103). We note that for $p_1>p_2$ the class of domains $\{D^{(p1)}\}$ contains the class of domains $\{D^{(p2)}\}$, but the integral representations become more complicated with increasing p. Therefore, it turns out that for domains of class (T) the method we are using cannot be used to obtain integral representations of simpler form than Temljakov's (4.103).

b) We choose the domain $D_0\{|z_1|+|z_2|<1\}$ (this domain is sometimes called a hypercone). Setting $\mu_0(|\zeta|)=d|\zeta_1|$, we find that in this case

$$h_0(\mathfrak{z})=\frac{(1-\mathfrak{z}_1-\mathfrak{z}_2)(1+2\mathfrak{z}_1\mathfrak{z}_2-\mathfrak{z}_1^2-\mathfrak{z}_2^2)+8\mathfrak{z}_1\mathfrak{z}_2}{[1+\mathfrak{z}_1^2+\mathfrak{z}_2^2-2\mathfrak{z}_1-2\mathfrak{z}_2-2\mathfrak{z}_1\mathfrak{z}_2]^2}$$

and we obtain the following integral representation for the domain D_0:

$$f(z)=\frac{1}{(2\pi i)^2}\int_0^1 d|\zeta_1|\int_{\Delta_{|\zeta|}}\frac{(1-\mathfrak{z}_1-\mathfrak{z}_2)(1+2\mathfrak{z}_1\mathfrak{z}_2-\mathfrak{z}_1^2-\mathfrak{z}_2^2)+8\mathfrak{z}_1\mathfrak{z}_2}{[1+\mathfrak{z}_1^2+\mathfrak{z}_2^2-2\mathfrak{z}_1-2\mathfrak{z}_2-2\mathfrak{z}_1\mathfrak{z}_2]^2}f(\zeta)\,\frac{d\zeta}{\zeta}.$$

Here $\mathfrak{z}_1=z_1\bar{\zeta}_1$, $\mathfrak{z}_2=z_2\bar{\zeta}_2$.

We now consider the domain $D\{|z_2|<\Phi(|z_1|)\}$, where Φ is an arbitrary function twice continuously differentiable, satisfying for $0\le|z_1|\le r$ the condition

$$\Phi(|z_1|)\Phi'(|z_1|)+2|z_1|\Phi(|z_1|)\Phi''(|z_1|)-|z_1|\Phi'^2(|z_1|)<0.$$

This condition is automatically satisfied when the domain D is convex.

We choose

$$\phi_1(|\zeta_1|) = \frac{[|\zeta_1|\,\Phi'(|\zeta_1|)]^2}{[|\zeta_1|\,\Phi'(|\zeta_1|) - \Phi(|\zeta_1|)]^2},$$

$$\phi_2(|\zeta_2|) = (1 - \sqrt{\phi_1})^2, \quad \mu(|\zeta|) = \frac{1}{2}\,\phi_1'\phi_1^{-\frac{1}{2}}\,d|\zeta_1|.$$

It may be shown that all conditions of Theorem 23.4 will be satisfied in this case, and we obtain for the domain D the integral representation (4.92′) with the kernel $h(\mathfrak{z}) = h_0(\mathfrak{z})$. It has the form

$$f(z) = \frac{1}{2(2\pi i)^2} \int_0^r \frac{\phi_1'(|\zeta_1|)}{\sqrt{\phi_1(|\zeta_1|)}}\,d|\zeta_1| \times$$

$$\times \int_{\Delta_{|\zeta|}} \frac{(1 - \mathfrak{z}_1 - \mathfrak{z}_2)(1 + 2\mathfrak{z}_1\mathfrak{z}_2 - \mathfrak{z}_1^2 - \mathfrak{z}_2^2) + 8\mathfrak{z}_1\mathfrak{z}_2}{[1 + \mathfrak{z}_1^2 + \mathfrak{z}_2^2 - 2\mathfrak{z}_1 - 2\mathfrak{z}_2 - 2\mathfrak{z}_1\mathfrak{z}_2]^2}\, f(\zeta)\,\frac{d\zeta}{\zeta},$$

where

$$\mathfrak{z}_1 = z_1\bar{\zeta}_1\,\frac{\phi_1(|\zeta_1|)}{|\zeta_1|^2}, \quad \mathfrak{z}_2 = z_2\bar{\zeta}_2\,\frac{\phi_2(|\zeta_2|)}{|\zeta_2|^2},$$

$$|\zeta_2| = \Phi(|\zeta_1|), \quad 0 \le |\zeta_1| \le r.$$

c) Theorems 23.1–23.4 can be extended to certain classes of non-bounded n-circular domains. In this direction, for example, it is possible to obtain for the bicircular domain $D_0\{|z_2|^2 < 2|\ln|z_1||\}$ an integral representation

$$f(z) = \frac{2}{(2\pi i)^2} \int_0^1 |\zeta_1|\,d|\zeta_1| \int_{\Delta_{|\zeta|}} \frac{e^{z_2\bar{\zeta}_2} f(\zeta)}{(1 - z_1\bar{\zeta}_1 e^{z_2\bar{\zeta}_2})^2}\,\frac{d\zeta}{\zeta}.$$

Proceeding exactly as in the preceding examples, we may extend this representation to a certain class of n-circular domains.

We now indicate a generalization of Theorems 23.1–23.4. We shall call the set $\vartheta \subset \partial D^+$ *bounding for monomials* in the domain D if to every vector k there corresponds a point $|\zeta| \in \vartheta$ such that $|\zeta|^k = d_k(D)$. We assume that there exists a finite or countable set of piecewise smooth surfaces $\vartheta_p \subset \partial D^+$, $p = 1, 2, \cdots$,

where $1 \leq \operatorname{Dim} \vartheta_p \leq n-1$, such that the set $\bigcup_p \vartheta_p$ is bounding for monomials in the domain D. We divide the set of vectors k into sets N_p $(p = 1, 2, \cdots)$ so that if $k \in N_p$ we have $\sup |\zeta|^k = d_k(D)$. On every torus $\Delta_{|\zeta|}$, where $|\zeta| \in \vartheta_p$ we define a continuous function $\nu_p(\theta)$, where $\theta = (\theta_1, \cdots, \theta_n)$, $\theta_j = \zeta_j/|\zeta_j|$, $j = 1, \cdots, n$, such that for a suitable orientation of this torus and for all $\tilde{k} \in N_p$ and for arbitrary k

$$\frac{1}{(2\pi i)^n} \int_{\Delta_{|\zeta|}} \zeta^k \bar{\zeta}^{\tilde{k}} \nu_p(\theta) \frac{d\zeta}{\zeta} = |\zeta|^k |\zeta|^{\tilde{k}} \delta_{k,\tilde{k}}.$$

On each of the surfaces ϑ_p we define a continuous odd form $\mu_p(|\zeta|)$ of degree $\operatorname{Dim}(\vartheta_p)$ and a sequence of continuous functions $\{\psi_k(|\zeta|)\}$. Choosing the form $\mu_p(|\zeta|)$ and the function $\psi_k(|\zeta|)$ so that

$$b_k = [\int_{\vartheta_p} \psi_k(|\zeta|)\mu_p(|\zeta|)]^{-1} < \infty, \quad k \in N_p,$$

we consider the function

$$H_p(z, \zeta) = \sum_{k \in N_p} z^k \bar{\zeta}^k \frac{\psi_k(|\zeta|)}{|\zeta|^{2k}} b_k, \tag{4.118}$$

defined in the domain of convergence of the series (4.118). Then we have

THEOREM 23.8 (Aĭzenberg [4]). *If*

1) *the series* (4.118) *converges for all* $z \in D$, $\zeta \in \vartheta_p$;

2) *at every point* $z \in D$ *the series* (4.118) *converges in the mean with respect to repeated integration over the surfaces* $\Delta_{|\zeta|}$ *and* ϑ_p (*the meaning of this condition is similar to that of condition* 2) *of Theorem* 23.1);

3) *the function* $f(z)$ *is holomorphic in the domain* D *and continuous on the closed domain* $\bar{D}$;

then in the domain D *we have the integral representation*

$$f(z) = \frac{1}{(2\pi i)^n} \sum_p \int_{\vartheta_p} \mu_p(|\zeta|) \int_{\Delta_{|\zeta|}} f(\zeta) H_p(z, \zeta) \nu_p(\theta) \frac{d\zeta}{\zeta}. \tag{4.119}$$

The series (4.119) *converges absolutely and uniformly in the domain* D.

Theorems 23.2–23.4 may be generalized in a similar way.

The results we have obtained may be extended to the case when some or even all of the surfaces ϑ_p reduce to distinct points. In the later case, by a

suitable choice of the functions $\nu_p(\theta)$, the integral representation (4.119) leads to the integral representation of de Leeuw [1]; if the n-circular domain D is an analytic polyhedron, we may use this process to obtain a Weil integral representation for an n-circular Weil domain.

We note that the results of subsections 1 and 3 of this section are still valid if in place of the form $\mu(|\zeta|)$ we use an arbitrary finite measure.[1)] Theorems 23.1–23.3 may be extended to other types of circular domains, in particular to the $(p_1, \cdots, p_n)$-circular domains.

4. Integral representations with a Szegö kernel in tubular domains. We agree to say that a tubular domain S in the space C^n of the variables $z_k = x_k + iy_k$, $k = 1, \cdots, n$, belongs to the class (B) if its base: B_1) $S^{(\mathrm{Im})}$ is convex; B_2) does not contain whole lines; B_3) is bounded by the piecewise smooth hypersurface $\partial S^{(\mathrm{Im})}$.

Suppose the tubular domain $S \in (B)$. Let O be the origin of coordinates in the space R_n of the variables $y_1, \cdots, y_n$, and let the hyperplanes $\{\Gamma\}$ pass through it and lie parallel to the limiting positions of the tangent hyperplanes of the hypersurface $\partial S^{(\mathrm{Im})}$ (these directions are defined by allowing the points of tangency of the tangent hyperplanes of the hypersurface $\partial S^{(\mathrm{Im})}$ to recede to infinity in all possible ways). We consider the convex cone with vertex at the origin enveloped by these hyperplanes. The nappe of this cone lying on the same side of the hyperplanes $\{\Gamma\}$ as the domain $S^{(\mathrm{Im})}$ (this side is determinate, by condition B_2) is called the *asymptotic* cone of the domain S. As the asymptotic cone of a domain S with a bounded base $S^{(\mathrm{Im})}$ we take the set $\{O\}$, namely, the origin of coordinates.

If V is the asymptotic cone of the domain S, we say that this domain S is of type V. Here, as usual, a convex cone W_0 with vertex at the origin of coordinates in the space R_n of the variables $y_1, \cdots, y_n$ is a domain with the following properties: if the points $(y_1, \cdots, y_n)$, $(y'_1, \cdots, y'_n) \in W_0$, then all points $(\lambda y_1 + \mu y'_1, \cdots, \lambda y_n + \mu y'_n) \in W_0$, where $\lambda, \mu \geq 0$, but the point $(0, \cdots, 0) \notin W_0$. A convex cone with vertex at an arbitrary point of the space R_n is obtained by parallel displacement of the cone W_0. In what follows we shall consider only convex cones and will not take special note, as a rule, of this fact. A cone is said to be *non-degenerate* if it contains some topologically n-dimensional sphere.

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1) See Aĭzenberg [4,6].

Let W be a cone with vertex at the origin of coordinates. The points $(y_1^*, \cdots, y_n^*) \in R_n$, satisfying the condition $\sum_{j=1}^n y_j y_j^* > 0$ for all points $(y_1, \cdots, y_n) \in \overline{W}$, $(y_1, \cdots, y_n) \neq 0$, form the *conjugate* cone W^*. If the cone W does not contain entire lines, the conjugate cone W^* is non-degenerate. If $W = \{O\}$ we set $W^* = R_n$. A tubular domain S is said to be *W-shaped* if the domain $S^{(\mathrm{Im})}$ contains, with every point y_0, the closed cone $\overline{W}_{y_0}$ obtained from the closed cone W with vertex at the origin of coordinates by parallel displacement such that the origin goes into the point y_0. We remark that the octant-shaped domain S is a particular kind of W-shaped domain. In this case the cone W is the octant $\{y_1 > 0, \cdots, y_n > 0\}$. The simplest class of W-shaped tubular domains are the *radial* domains; the base of a radial domain is the cone W itself (we always take this with vertex at the origin of coordinates). An example of a tubular radial domain is given by the product of the half-planes $S_1\{\operatorname{Im} z_1 > 0, \cdots, \operatorname{Im} z_n > 0\} \subset C^n$. The base of such a domain is the octant $V_1\{y_1 > 0, \cdots, y_n > 0\} \subset R_n$. For $n = 2$ the basis of a radial domain is an angle in the plane $x_1 = x_2 = 0$ with vertex at the origin.

If $W_1, W_2 \subset R_n$ are cones with vertex at the origin of coordinates and $W_1 \subset W_2$, then every W_2-shaped tubular domain is also W_1-shaped.

A tubular domain $S \in (B)$ of type V is always V-shaped; and V is the maximal cone with respect to which the domain S has this property. We remark that by condition B_2 the cone V^* conjugate to the cone V is non-degenerate.

Now we pass on to the development of integral representations for tubular domains. In the calculation of the Szegö kernel for n-circular domains an essential role was played by the decomposition of a function holomorphic on such a domain into a power series. This arises from the fact that the function $z^k = z_1^{k_1} \cdots z_n^{k_n}$ is multiplied by a number with absolute value unity under the mappings (2.58) (the center of the n-circular domain is translated to the origin of coordinates) which form a group of automorphisms of the domain. The functions $e^{i\|\lambda z\|}$ play an analogous role in the theory of tubular domains, with the usual meanings $\|\lambda z\| = \lambda_1 z_1 + \cdots + \lambda_n z_n$ and $(\lambda_1, \cdots, \lambda_n) \in R_n$. These functions do not form a discrete set; therefore the expansion into a power series must be replaced by an expansion in integrals (by Fourier integrals). In contrast to the n-circular domains, the various classes of tubular domains are taken care of by functions $e^{i\|\lambda z\|}$ with various sets of values λ_k.

We consider the functions $e^{i\|\lambda z\|}$, where $z \in S$, $\lambda \in R_n$. Then

$|e^{i\|\lambda z\|}| = e^{-\|\lambda y\|}$. It can be shown that in the domain S such a function is bounded if and only if $\lambda \in V^*$. Suppose that $M(\lambda) = \sup_{z\in S}|e^{i\|\lambda z\|}|$, where $\lambda \in V^*$. Obviously, each function $e^{i\|\lambda z\|}$ can attain the corresponding value $M(\lambda)$ only on the boundary of the domain S. Denote by Ω_S the closure of the set of boundary points of the domain S at which the function $e^{i\|\lambda z\|}$ attains its maximum value; we call Ω_S the skeleton of the boundary of the domain S. It is obvious that the skeleton Ω_S is a tubular set (a tubular set and the base of a tubular set are defined in the same way as for a tubular domain). We assume that the base of the skeleton $\Omega_S^{(\mathrm{Im})}$ of the domain S is a surface of class $\mathcal{C}^1$ of topological dimension $k < n$. The base of a radial tubular domain is a plane $\{y_1 = 0, \cdots$ $\cdots, y_m = 0\}$. We shall say that a continuous odd form ψ of degree $k < n$, defined on the surface $\Omega_S^{(\mathrm{Im})}$, belongs to the class (Γ) if it vanishes only at isolated points and if

$$c(\lambda) = (2\pi)^n \int_{\Omega_S^{(\mathrm{Im})}} e^{-2\|\lambda y\|}|\psi| < \infty \text{ for } \lambda \in V^*. \tag{4.120}$$

The symbol $|\psi|$ means that after a transformation under the integral sign to the coordinates of the surface $\Omega_S^{(\mathrm{Im})}$, the function obtained from the form ψ is replaced by its absolute value. Since ψ is an odd form, the choice of coordinate system on the surface $\Omega_S^{(\mathrm{Im})}$ is not essential to the formulation of our condition. For a radial domain S we set $c(\lambda) = (2\pi)^n$, $\psi = 1$.

We agree that a function f belongs to the class $F(\psi)$ (where $\psi \in (\Gamma)$) in the domain $S \in (B)$ if it is holomorphic in the domain S and continuous on the closed domain $\bar{S}$, and if

$$\int_{\Omega_S} |f|^2 |\psi \wedge dx_1 \wedge \cdots \wedge dx_n| < \infty. \tag{4.121}$$

The *Szegö kernel* of a domain $S \in (B)$ connected with a form $\psi \in (\Gamma)$ is a function $K(w, z)$ such that: 1) for $w \in S$, $z \in \bar{S}$ it is holomorphic in the variables $w_1, \cdots, w_n$, $\bar{z}_1, \cdots, \bar{z}_n$; 2) for every function $f \in F(\psi)$ and arbitrary $w \in S$ we have

$$f(w) = \int_{\Omega_S} K(w, z) f(z) \psi \wedge dx_1 \wedge \cdots \wedge dx_n. \tag{4.122}$$

If the function $K(w, z)$, for $w \in S$, $z \in \bar{S}$, is holomorphic in the variables

$w_1, \cdots, w_n$ and continuous in the variables $z_1, \cdots, z_n$, and satisfies condition 2), it is called a *generalized Szegö kernel* for the domain S connected with the form ψ. We have [1)]

THEOREM 23.9. *Suppose that the tubular domain $S \in (B)$ and that the form $\psi \in (\Gamma)$. Then there exists one and only one Szegö kernel for the domain S connected with the form ψ. It is defined by the equation*

$$K(w, z) = \int_{V^*} \frac{1}{c(\lambda)} e^{i\|\lambda(w-\bar{z})\|} d\lambda_1 \cdots d\lambda_n. \tag{4.123}$$

The integral converges uniformly for $z \in \bar{S}$, $w \in S'$. Here S' is an arbitrary tubular domain lying in S and satisfying the condition $r(\zeta, \omega) > \epsilon$, where $\zeta \in \partial S$, $\omega \in \partial S'$, $r(\omega, \zeta)$ is the distance between the points ζ and ω, and $\epsilon > 0$ is an arbitrary number.

5. **Examples of integral representations in tubular domains.** For certain classes of domains S the integral (4.123) can be put into explicit form. The most important case is that of a homogeneous tubular domain $S \in (B)$, when for every pair of points of the domain S there exists an automorphism of the domain carrying one of the points into the other. For some types of radial homogeneous domains the kernel $K(w, z)$ was found by Bochner. [2)] The more general case has been considered by Gindikin. [3)] If the domain S is inhomogeneous but nevertheless admits a sufficiently large group of automorphisms, the kernel $K(w, z)$ satisfies certain definite relationships by means of which it can be computed.

In computing kernels we shall make use of the fact that the kernel $K(w, z)$ is a holomorphic function of the quantities $w_1 - \bar{z}_1, \cdots, w_n - \bar{z}_n$. Thus it is easy to see that the function need only be written out for $w = z$, that is, we must find

$$K(z, z) = \int_{V^*} \frac{1}{c(\lambda)} e^{-2\|\lambda y\|} d\lambda_1 \cdots d\lambda_n \text{ for } y \in S^{(\mathrm{Im})}, \tag{4.123$_1$}$$

and then replace y_k by $(w_k - \bar{z}_k)/2$, $k = 1, \cdots, n$.

1) The base of the radial domain $S_1 \subset C^n$ is the octant

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1) See Gindikin [3]. Theorem 23.9 was proved earlier by Bochner [2] in a less general form.

2) See Bochner [2], Bochner-Martin [1], Chapter VI (where one may also find further references).

3) See Gindikin [1,2].

$V_1\{y_1 > 0, \cdots, y_n > 0\} \subset R_n$. In this case $V_1^* = V_1$ and

$$K(z, z) = \frac{1}{(2\pi)^n} \int_0^\infty \cdots \int_0^\infty e^{-2(\lambda_1 y_1 + \cdots + \lambda_n y_n)}\, d\lambda_1 \cdots d\lambda_n =$$

$$= [(2\pi i)^n (z_1 - \bar{z}_1) \cdots (z_n - \bar{z}_n)]^{-1}.$$

Hence

$$K(w, z) = \frac{1}{(2\pi i)^n} \frac{1}{(w_1 - \bar{z}_1) \cdots (w_n - \bar{z}_n)}. \tag{4.124}$$

2) The base of the radial domain $S_2 \subset C^3$ is the cone $V_2\left\{\begin{vmatrix} y_1 & y_2 \\ y_2 & y_3 \end{vmatrix} > 0,\ y_1 > 0\right\} \subset R_3$ (the cone of a symmetric positive definite second-order matrix). The cone V_2^* consists of the points $(\lambda_1, \lambda_2, \lambda_3)$ for which the matrix $\begin{bmatrix} \lambda_1 & \frac{1}{2}\lambda_2 \\ \frac{1}{2}\lambda_2 & \lambda_3 \end{bmatrix}$ is positive definite. Then

$$K(z, z) = \frac{1}{(2\pi)^3} \int_{V_2^*} e^{-2(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3)}\, d\lambda_1 d\lambda_2 d\lambda_3 =$$

$$= \frac{i}{8\pi^3} \begin{vmatrix} z_1 - \bar{z}_1 & z_2 - \bar{z}_2 \\ z_2 - \bar{z}_2 & z_3 - \bar{z}_3 \end{vmatrix}^{-\frac{3}{2}}.$$

Hence

$$K(w, z) = \frac{i}{8\pi^3} \begin{vmatrix} w_1 - \bar{z}_1 & w_2 - \bar{z}_2 \\ w_2 - \bar{z}_2 & w_3 - \bar{z}_3 \end{vmatrix}^{-\frac{3}{2}}. \tag{4.125}$$

The function (4.125) has no branch points for $w \in S_2$, $z \in \bar{S}_2$. Therefore it is possible to distinguish a single-valued branch by evaluating it at $w = z$.

3) The base of the domain $S_3 \subset C^3$ is the domain $\{y_1 y_2 > 1,\ y_1 > 0\} \subset R_2$.

This domain has the type $V_1 \subset R_2$ (from the first example). Its skeleton coincides with the entire boundary, and therefore $\partial S_3 = \Omega_{S_3}\{y_2 = 1/y_1,\ y_1 > 0\} \subset C^3$. We set $\psi = dy_1/\sqrt{y_1}$. Then

$$c(\lambda_1, \lambda_2) = (2\pi)^2 \int_{\Omega_{S_3}^{(\mathrm{Im})}} e^{-2(\lambda_1 y_1 + \lambda_2 y_2)} \frac{dy_1}{\sqrt{y_1}} = \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{\lambda_1}} e^{-4\sqrt{\lambda_1 \lambda_2}},$$

$$K(z, z) = 2^{-\frac{3}{2}}\pi^{-\frac{5}{2}}\int_0^\infty\int_0^\infty \sqrt{\lambda_1}\, e^{4\sqrt{\lambda_1\lambda_2}-2(\lambda_1 y_1+\lambda_2 y_2)}\, d\lambda_1 d\lambda_2 =$$

$$= [64\pi^2\sqrt{y_1}(1-\sqrt{y_1 y_2})^2]^{-1}.$$

Hence

$$K(w, z) = \left[16\pi^2\sqrt{\frac{2}{i}(w_1-\bar{z}_1)}(2+i\sqrt{(w_1-\bar{z}_1)(w_2-\bar{z}_2)})^2\right]^{-1}. \tag{4.126}$$

4) The base of the domain $S_4 \subset C^2$ is the domain $\{y_2 > e^{-y_1}\}$; it is also of type $V_1 \subset R_2$ (from the first example). Its skeleton coincides with the entire boundary, and therefore $\partial S_4 = \Omega_{S_4}\{y_2 = e^{-y_1}\}$. We set $\psi = dy_1$. Then

$$c(\lambda_1, \lambda_2) = (2\pi)^2(2\lambda_2)^{-2\lambda_1}\Gamma(2\lambda_1),$$

$$K(w, z) = \left[8\pi^2\left(\frac{w_2-\bar{z}_2}{i}\right)\left(\ln\frac{w_2-\bar{z}_2}{2i}+\frac{w_1-\bar{z}_1}{2i}\right)^2\right]^{-1}. \tag{4.127}$$

5) The base of the domain $S_5 \subset C^2$ is the domain $\{y_2 > 2y_1^2\}$; it has the type $V\{y_1 = 0,\ y_2 > 0\} \subset R_1$; therefore the asymptotic cone of this domain is degenerate. Its skeleton coincides with the entire boundary and accordingly $\Omega_{S_5} = \partial S_5\{y_2 = 2y_1^2\}$. The cone conjugate to the ray V_5 is the half-plane $V^*\{\lambda_2 > 0\}$. We set $\psi = dy_1$. Then

$$c(\lambda_1, \lambda_2) = 2\pi^{\frac{5}{2}}(\lambda_2)^{-\frac{1}{2}}\exp\left(\frac{\lambda_1^2}{4\lambda_2^2}\right),$$

$$K(w, z) = \pi^{-2}[(w_1-\bar{z}_1)^2 - i(w_2-\bar{z}_2)]^{-2}. \tag{4.128}$$

6) The base of the domain $S_6 \subset C^3$ is the domain

$$\left\{\begin{vmatrix} y_1 & y_2 \\ y_2 & y_3\end{vmatrix} > 1,\ y_1 > 0\right\}.$$

It has the type V_2. Its skeleton coincides with the entire boundary; the cone V_2^* is described in the second example. We set $\psi = |(1/y_1)\, dy_1 \wedge dy_2|$. Then

$$c(\lambda_1, \lambda_2, \lambda_3) = 2\pi^4(4\lambda_1\lambda_3 - \lambda_2^2)^{-\frac{1}{2}} \exp[-2\sqrt{4\lambda_1\lambda_2 - \lambda_2^2}],$$

$$K(w, z) = 2\pi^{-3}[(w_2 - \bar{z}_2)^2 - (w_1 - \bar{z}_1)(w_3 - \bar{z}_3)]^{-\frac{1}{2}} \times$$
$$\times [\sqrt{(w_2 - \bar{z}_2)^2 - (w_1 - \bar{z}_1)(w_3 - \bar{z}_3)} - 1]^{-3}. \tag{4.129}$$

6. **Integral representations with generalized Szegö kernel in tubular domains.** We have

THEOREM 23.10 (Gindikin [3]). *Suppose that the tubular domain $S \in (B)$ and the form $\psi \in (\Gamma)$. If for the real-valued functions $\phi_1, \cdots, \phi_n$ belonging to the class $\mathcal{C}^1$ we have on the base $S^{(\mathrm{Im})}$*

$$d(\lambda) = (2\pi)^n \int_{\Omega_S^{(\mathrm{Im})}} e^{-2\|\lambda\phi\|}\,|\psi| < \infty \quad \text{for } \lambda \in V^*,$$

$$\sup_{y \in S^{(\mathrm{Im})}} \|\lambda(y - \phi)\| = \sup_{y \in S^{(\mathrm{Im})}} (-\|\lambda\phi\|) - \sup_{y \in S^{(\mathrm{Im})}} (-\|\lambda y\|),$$

then the function

$$K(w, z) = \int_{V^*} \frac{1}{d(\lambda)} e^{\|i\lambda(w-z)-\phi\|}\, d\lambda_1 \cdots d\lambda_n \tag{4.130}$$

is a generalized Szegö kernel for the domain S_1 connected with the form ψ. Under the given conditions, the integral (4.130) converges uniformly for $z \in \bar{S}$, $w \in S'$ (the domain S' is defined as in Theorem 23.9).

We remark that the generalized Szegö kernel is not uniquely defined for the domain S and the form ψ.

EXAMPLE. Let the base $S_7^{(\mathrm{Im})}$ of the tubular domain $S_7 \subset C^2$ be given by the condition $y_2 > \chi(y_1)$, where $\chi \in \mathcal{C}^2$ for all values y_1 and $\chi''(y_1) \geq 4$. Then the function

$$K(w, z) = \pi^{-2}\left[\left(w_1 - z_1 + i\,\frac{\chi'(y_1)}{2}\right)^2 - i\left(w_2 - z_2 + i\,\frac{(\chi'(y_1))^2}{4}\right)\right]^{-2} \tag{4.131}$$

is the generalized Szegö kernel for this domain connected with the form $\psi = (1/4)\chi''(y_1)\,dy_1$. The integral representation (4.122) with this kernel is analogous to the Temljakov integral of the second kind for the bicircular domains. For $\chi(y_1) = 2y_1^2$ the form $\psi = dy_1$, and the domain S_7 reduces to the domain S_5, the kernel (4.131) to the kernel (4.128).

CHAPTER V

FUNCTIONS MEROMORPHIC IN THE WHOLE SPACE C^n. ENTIRE FUNCTIONS

This chapter is devoted to a study of functions that are meromorphic in the whole space C^n, and to a study of entire functions in particular. As in the classical case of one variable, a function holomorphic at all points of the space C^n will be called an entire function.

§ 24. FUNCTIONS MEROMORPHIC IN THE EXTENDED SPACE

1. Strengthening of Liouville's theorem. Theorem 21.2, which deals with the possibility of analytic extension of functions that are holomorphic on the boundary of some domain to functions inside the domain (when the boundary is connected), allows us to prove the following assertion, which is a strengthening of Liouville's theorem. We formulate the theorem, and prove it, for the projectively completed space P^n. It holds, as do the other assertions of this section, for the extended space of function theory as well.

THEOREM 24.1. *A function $f(\zeta)$, where $\zeta \in P^n$, which is holomorphic at all points of some $(n-1)$-complex-dimensional complex projective plane of the space P^n is a constant.*

PROOF. By means of the projective transformation (1.76) we transfer the given plane to infinity in the space $P^n_{\zeta'}$. Then the function $F(\zeta') \equiv f(\zeta)$ is holomorphic and bounded at all points of the space $C^n_{z'} \subset P^n_{\zeta'}$ of the variables $z'_1, \cdots, z'_n$, for which $\sum_{k=1}^n |z'_k|^2 \geq R^2$. Here R is a sufficiently large number. Since the function $F(\zeta')$ is holomorphic at the points of the hypersphere

$$\sum_{k=1}^{n} |z'_k|^2 = R^2,$$

it follows from Theorem 21.2 that it can be analytically continued to the whole interior of this hypersphere. Then by Liouville's theorem we conclude that the function $F(\zeta')$ is constant, and therefore so is $f(\zeta)$.

2. The Weierstrass-Hurwitz theorem. From this result we now derive an important theorem, stated by Weierstrass and proved by Hurwitz. It is analogous to the well-known theorem of classical analysis on rational functions.

THEOREM 24.2 (Weierstrass-Hurwitz). *A function $f(\zeta)$, where $\zeta \in P^n$, which is meromorphic at all points of the complex projective space P^n, is rational.*

PROOF. We prove the theorem for the space $P^2_\zeta \supset C^2_{w,z}$, where $C^2_{w,z}$ is the space of the complex variables w, z. We may assume that the function f is holomorphic at the infinitely distant point $(\infty, 0)$ and that $f(\infty, 0) \neq 0$. If the assumption were false, we could easily make it true by substituting for the function $f(w, z)$ the function

$$f\left[\frac{1}{\omega} + \alpha, \frac{\zeta}{\omega} + \beta\right] + C = \phi(\omega, \zeta),$$

where (α, β) is a point of holomorphy of the function $f(w, z)$ and C is a constant.

It is clear that the function resulting from this transformation is rational if and only if the initially given function is rational. Therefore we may immediately consider functions having the property indicated above. Then without loss of generality we may assume, first, that there exist two numbers r, $d > 0$ such that for $|w| > r$, $|z/w| < d$ the function $f(w, z)$ is holomorphic, and second, that in every plane $z = z_0$ there exists a finite set of zeros $(w_1, z_0), \cdots, (w_n, z_0)$ and of poles $(w^1, z_0), \cdots, (w^p, z_0)$ of this function.[1)]

For every zero (w_k, z_0) we may construct a pseudopolynomial

$$\phi_k(w, z) = (w - w_k)^{nk} + A_1^{(k)}(z)\,(w - w_k)^{nk-1} + \cdots + A_{nk}^{(k)}(z) \qquad (5.1_1)$$

such that the ratio $f(w, z)/\phi_k(w, z)$ does not vanish and is holomorphic at the point (w_k, z_0). Here the coefficients $A_\nu^{(k)}(z_0) = 0$, so that the function $1/\phi_k$ is holomorphic everywhere in the plane $z = z_0$ except at the point (w_k, z_0). We denote the product of all the functions ϕ_k in the plane $z = z_0$ by $\phi_{z_0}(w, z)$.

Similarly, for every pole (w^k, z_0) we may define a pseudopolynomial

1) Here n and p denote the number of zeros and poles, respectively, of the function $f(w, z)$ in the plane $z = z_0$, each zero or pole being counted according to its multiplicity for the function of one variable $f(w, z_0)$. We are concerned with zeros and poles lying in the finite portion of the plane.

$$\phi_k(w, z) = (w - w^k)^{pk} + B_1^{(k)}(z)(w - w^k)^{pk-1} + \cdots + B_{pk}^{(k)}(z) \qquad (5.1_2)$$

such that the function $f(w, z)\, \psi_k(w, z)$ will be holomorphic and different from zero at the point (w^k, z_0). The coefficients $B_\nu^{(k)}(z)$ are holomorphic for $z = z_0$ and $B_\nu^{(k)}(z_0) = 0$, so that the function ψ_k is holomorphic everywhere in the plane $z = z_0$ and vanishes nowhere except at the point (w^k, z_0). We denote the product of all the functions ψ for the plane $z = z_0$ by $\psi_{z_0}(w, z)$. Then the expression

$$f(w, z)\frac{\psi_{z_0}(w, z)}{\phi_{z_0}(w, z)} \qquad (5.2)$$

is holomorphic and non-vanishing for all values of z satisfying the condition $|z - z_0| < \epsilon(z_0)$ and for all finite values of w. We note further that if (w^*, z_0) is a point at which $f(w, z)$ is not defined, it supplies a factor in both the numerator and the denominator of the fraction in (5.2). Finally, we note that the polynomials (in w) $\phi_{z_0}(w, z)$ and $\psi_{z_0}(w, z)$ may be taken as relatively prime. They will have the form

$$\phi_{z_0}(w, z) = w^n + C_1^{(z_0)}(z)\, w^{n-1} + \cdots + C_n^{(z_0)}(z), \qquad (5.3)$$

$$\psi_{z_0}(w, z) = w^p + D_1^{(z_0)}(z)\, w^{p-1} + \cdots + D_p^{(z_0)}(z). \qquad (5.4)$$

If $|\zeta - z_0| < \epsilon(z_0)$, the quantities $C_\nu^{(z_0)}(\zeta)$, $D_\nu^{(z_0)}(\zeta)$ are elementary symmetric functions of the w-coordinates of the zeros and non-essential singular points of the function $f(w, z)$ in the plane $z = \zeta$ (in this connection every zero $w_k(\zeta)$ or pole $w^k(\zeta)$ is counted with the same multiplicity as for $f(w, \zeta)$: this multiplicity is defined by the solution of the equations $\phi_{z_0}(w, \zeta) = 0$, $\psi_{z_0}(w, \zeta) = 0$). It is clear that the position of these points does not depend on the choice of the center of the neighborhoods $|z - z_0| < \epsilon(z_0)$ that we are considering. Therefore $C_\nu^{(z_0)}(z)$, $D_\nu^{(z_0)}(z)$, $\phi_{z_0}(w, z)$, $\psi_{z_0}(w, z)$ do not depend on z_0; in what follows we will omit the subscript z_0 for the quantities C_ν, D_ν, ϕ, ψ.

In view of our condition we shall have for all roots $w(z)$ of the equations $\phi(w, z) = 0$ and $\psi(w, z) = 0$ either $|w| < r$ or $|w| < |z|/d$. Therefore for their elementary symmetric functions we have the inequalities

$$|C_\nu(z)| < \alpha \cdot \max(r^n, |z|^n); \quad |D_\nu(z)| < \alpha \cdot \max(r^p, |z|^p) \qquad (5.5)$$

(here α is some constant).

It follows from this that the coefficients $C_\nu(z)$, $D_\nu(z)$ are entire rational functions of the variable z; therefore $\phi(w, z)$ and $\psi(w, z)$ are entire rational

functions of the variables w, z. Since they fulfill their role for all finite values of z, the function $F(w, z) = f(w, z)\,\psi(w, z)/\phi(w, z)$ will be holomorphic and different from zero everywhere in the finite portion of the space.

If this function is not a constant, it must have a pole at an infinitely distant point of the space P^2. Such a pole may be taken as an infinitely distant point of the plane $w = az$. But then the function $1/F(w, z)$ is holomorphic in this extended analytic plane; then (by the preceding theorem) it must be constant. With this our theorem is proved.

We examine two corollaries of this theorem.

COROLLARY 1. *A function $f(\zeta)$, where $\zeta \in P^n$, which is meromorphic at all points of some $(n-1)$-complex-dimensional complex projective plane is rational.*

COROLLARY 2. *An entire non-rational function has all infinitely distant points of the space P^n as essential singularities.*

The first of these corollaries is proved in the same way as the first theorem of this section. Instead of using Theorem 21.2 we must make use, however, of the corresponding theorem for meromorphic functions, and instead of Liouville's theorem we must use the Weierstrass-Hurwitz theorem. The second corollary is easily obtained from the first.

In conclusion, we prove two theorems that are useful in applications.

THEOREM 24.3. *A rational function $R(z_1, \cdots, z_n)$, which depends in an essential way on $k \geq 2$ variables has at least one ambiguous point.*

PROOF. If $R(z)$ is a polynomial, such a point of ambiguity must exist at infinity. Otherwise the function $1/R(z)$ would be holomorphic at infinity and by Theorem 24.1 would be a constant. It follows that there exists at infinity a point of ambiguity for the function $R(z) = 1/H(z)$, where $H(z)$ is a polynomial. This stems from the fact that every point of ambiguity of the function $H(z)$ is also a point of ambiguity of the function $(\alpha H + \beta)/(\gamma H + \delta)$ $(\alpha\delta - \beta\gamma \neq 0)$. It remains to consider the case when $R(z) = G(z)/H(z)$, where $G(z)$ and $H(z)$ are relatively prime non-constant polynomials. In this case the points of ambiguity of the function $R(z)$ will be the common roots of the equations $G(z) = 0$ and $H(z) = 0$. Since we are considering a complex domain, such points will necessarily exist for $k \geq 2$, either in the finite portion of the space or at infinity. This completes the proof of our theorem.

The other theorem is as follows:

THEOREM 24.4. *The most general meromorphic (in the strict sense) mapping of the whole projectively completed space P^n into itself has the form*

$$Z_i = \frac{a_{i0} + a_{i1}z_1 + \cdots + a_{in}z_n}{b_0 + b_1 z_1 + \cdots + b_n z_n}, \quad i = 1, \cdots, n, \tag{5.6}$$

that is, it is a projective mapping.

PROOF. The proof of this theorem will also be carried out for the space $P^2_\zeta \supset C^2_{w,z}$. If $w_1 = f(w, z)$, and $z_1 = \phi(w, z)$ are mappings of the projectively completed space P^2 which satisfy the conditions stated in the theorem, then by the Weierstrass-Hurwitz theorem both functions $f(w, z)$ and $\phi(w, z)$ are rational. We introduce homogeneous coordinates, writing $w = \zeta_1/\zeta_3$, $z = \zeta_2/\zeta_3$; then

$$\left.\begin{aligned} f(w, z) &= \frac{B(w, z)}{A(w, z)} = \frac{\beta(\zeta_1, \zeta_2, \zeta_3)}{\alpha(\zeta_1, \zeta_2, \zeta_3)}; \\ \phi(w, z) &= \frac{C(w, z)}{A(w, z)} = \frac{\gamma(\zeta_1, \zeta_2, \zeta_3)}{\alpha(\zeta_1, \zeta_2, \zeta_3)}. \end{aligned}\right\} \tag{5.7}$$

Here A, B, and C are polynomials having no common factor, and α, β, and γ are homogeneous polynomials of some degree m which have no common factor. By an immediate calculation we may show that

$$\frac{\partial(f, \phi)}{\partial(w, z)} = \frac{1}{mA^3\zeta_3^{3(m-1)}} \frac{\partial(\alpha, \beta, \gamma)}{\partial(\zeta_1, \zeta_2, \zeta_3)}. \tag{5.8}$$

By the conditions stated in the theorem, the Jacobian $\partial(f, \phi)/\partial(w, z)$ cannot vanish at any point of the space. It follows that the Jacobian $\partial(\alpha, \beta, \gamma)/\partial(\zeta_1, \zeta_2, \zeta_3)$ is also different from zero at all points of the space. But it is a homogeneous polynomial of degree $3(m-1)$. Therefore $m = 1$, and therefore α, β, and γ are linear functions of ζ_1, ζ_2, ζ_3. This proves our theorem.

§25. COUSIN'S THEOREM

1. Cousin's first theorem for the space C^n. Almost the earliest work in the theory of analytic functions of several complex variables was devoted to the problem of extending to this theory the classical theorems of Mittag-Leffler and Weierstrass on the construction of a meromorphic function with preassigned poles and of an entire function with preassigned zeros. In 1894 Cousin [1] published a generalization of the Mittag-Leffler theorem on the determination of a meromorphic function by specification of its principal parts.

THEOREM 25.1 (first theorem of Cousin). *Suppose that to every point P of the space C^n (polycylindrical domain) there corresponds a neighborhood V_P of P and a function f_P meromorphic in that neighborhood. Suppose also that if two such neighborhoods V_P and V_Q of the points P and Q have a common portion, then the corresponding functions f_P and f_Q are, on the common portion, equivalent with respect to subtraction. Then there exists a function F meromorphic at all points of the space C^n (polycylindrical domain) and equivalent, with respect to subtraction, at each point P to the corresponding function f_P.*

EXPLANATION. Two functions both meromorphic at a point P are said to be equivalent with respect to subtraction at P if their difference is holomorphic at P.

PROOF. We carry the proof through in detail for the space C^n, and then indicate how it is to be modified for the case of the polycylindrical domain.

We denote our variables by $w_1, \cdots, w_{n-1}, z$. Let $S^{(k)}$ be the square in the w_k-plane defined by the inequalities $0 < \operatorname{Re}(w_k)$, $\operatorname{Im}(w_k) < R$ and let S_R be the square in the z-plane defined by the inequalities $0 < \operatorname{Re}(z)$, $\operatorname{Im}(z) < R$. We consider the closed polycylindrical domain $\overline{\Sigma}_R = \overline{S}_R^{(1)} \times \cdots \times \overline{S}_R^{(n-1)} \times \overline{S}_R$. We divide each of the squares $S_R^{(k)}$ and S_R into m^2 squares with side R/m. We number these squares and denote them by $q_\alpha^{(k)}$ and q_β. The number m may be chosen large enough so that every closed hypercube $\overline{q}_{\alpha_1}^{(1)} \times \cdots \times \overline{q}_{\alpha_{n-1}}^{(n-1)} \times \overline{q}_\beta$ is contained in some neighborhood V_p. In fact, if this were not true, then by repeated subdivision we could arrive at a point P_0 of the domain such that no hypercube with center at this point and sides parallel to the coordinate axes would be contained in any neighborhood V_p. It is sufficient, however, to choose the neighborhood V_{P_0} of the point P_0 and we immediately convince ourselves of the impossibility of finding a P_0 with the supposed property. Thus, the neighborhood we require can always be found; we denote it by $V_{\alpha_1 \cdots \alpha_{n-1}\beta}$. Now in every hypercube $\overline{q}_{\alpha_1}^{(1)} \times \cdots \times \overline{q}_{\alpha_{n-1}}^{(n-1)} \times \overline{q}_\beta$ the function f of our system is given.

Since $V_{\alpha_1 \cdots \alpha_{n-1}\beta}$ contains the closed domain $\overline{q}_{\alpha_1} \times \cdots \times \overline{q}_{\alpha_{n-1}} \times \overline{q}_\beta$, we have also that $V_{\alpha_1 \cdots \alpha_{n-1}\beta}$ contains some domain $Q_{\alpha_1}^{(1)} \times \cdots \times Q_{\alpha_{n-1}}^{(n-1)} \times \overline{q}_\beta$, where $Q_{\alpha_k}^{(k)}$ are squares with sides somewhat greater than R/m and situated in the same way as the $q_{\alpha_k}^{(k)}$. We choose two squares, q_{β_1}, q_{β_2} from S_R adjoining each other along the side $l_{\beta_1\beta_2}$. Let $f_{\beta_1}(w_1, \cdots, w_{n-1}, z)$ and $f_{\beta_2}(w_1, \cdots$ $\cdots, w_{n-1}, z)$ be meromorphic functions prescribed, according to the hypothesis of the theorem, in the neighborhoods $V_{\alpha_1 \cdots \alpha_{n-1}\beta_1}$ and $V_{\alpha_1 \cdots \alpha_{n-1}\beta_2}$. We consider the integral

$$l_{\beta_2\beta_1}(w_1, \cdots, w_{n-1}, z) = \frac{1}{2\pi i} \int_{l_{\beta_1\beta_2}} \frac{\phi(t)}{t-z}\, dt \tag{5.9}$$

(the side $l_{\beta_1\beta_2}$ is traversed in such a direction that q_{β_2} lies to the right of q_{β_1}). Here $\phi(z) = f_{\beta_2}(w_1, \cdots, w_{n-1}, z) - f_{\beta_1}(w_1, \cdots, w_{n-1}, z)$ (we do not need to display explicitly the dependence of the function ϕ on $w_1, \cdots, w_{n-1}$). This function is holomorphic at all points $(w_1, \cdots, w_{n-1}, z)$ for which the point $(w_1, \cdots, w_{n-1})$ lies in $Q^{(1)}_{\alpha_1} \times \cdots \times Q^{(n-1)}_{\alpha_{n-1}} = Q$, and z lies on the segment $\overline{l}_{\beta_1\beta_2}$. Then the function $\phi(z)$ will also be holomorphic in the domain $Q \times p_{3\rho}$, where $p_{3\rho}$ is a domain in the z-plane covered by disks of radius 3ρ with centers on $\overline{l}_{\beta_1\beta_2}$. Let z_0 be some point of the closed interval $\overline{l}_{\beta_1\beta_2}$ and r_{z_0} the disk of radius ρ with center at this point. If t and z are two points in the disk r_{z_0} then

$$\phi(t) = \phi(z) + (t-z)\,\phi'(z) + \frac{1}{2!}(t-z)^2\,\phi''(z) + \cdots \tag{5.10}$$

(the function $\phi(\zeta)$ is holomorphic in the disk of radius 2ρ with center at the point z; this disk always contains the point t: equation (5.10) follows from this).

We write

$$\psi(z, t) = \frac{\phi(t) - \phi(z)}{t-z}. \tag{5.11}$$

Suppose that the point $(w_1, \cdots, w_{n-1}, z)$ lies in $Q \times \overline{r}_{z_0}$. If the point t is fixed in the interval $l_{\beta_1\beta_2}$, outside the disk r_{z_0}, then it is clear that ψ is a holomorphic function of the variables $w_1, \cdots, w_{n-1}, z$ in $Q \times \overline{r}_{z_0}$. If the point t lies in a portion of the segment $\overline{l}_{\beta_1\beta_2}$ contained in r_{z_0}, then by (5.10)

$$\psi(z, t) = \phi'(z) + (t-z)\,\phi''(z) + \cdots. \tag{5.12}$$

This equation is used to extend the definition of the function $\psi(z, t)$ to the point $z = t$. Therefore the function $\psi(z, t)$ is: 1) a holomorphic function of the variables $w_1, \cdots, w_{n-1}, z$ in $Q \times \overline{p}_\rho$ for all $t \in \overline{l}_{\beta_1\beta_2}$; 2) a continuous function of the variables $w_1, \cdots, w_{n-1}, z, t$ for the values of these variables already prescribed (we make use of the fact that the point z_0 may occupy an arbitrary position on the interval $\overline{l}_{\beta_1\beta_2}$). Also

$$\phi(t) = \phi(z) + (t-z)\,\psi(z, t). \tag{5.13}$$

Substituting (5.13) into the integral (5.9) and denoting by z_1, z_2 the z-coordinates of the ends of the interval $l_{\beta_1\beta_2}$ (in the order defined above), we find that in $Q \times \overline{p}_\rho$ we have

$$I_{\beta_2\beta_1}(w_1, \cdots, w_{n-1}, z) = \frac{f_{\beta_2}(w_1, \cdots, w_{n-1}, z) - f_{\beta_1}(w_1, \cdots, w_{n-1}, z)}{2\pi i} \ln \frac{z_2 - z}{z_1 - z} + C(w_1, \cdots, w_{n-1}, z). \tag{5.14}$$

Here $C(w_1, \cdots, w_{n-1}, z)$ is a holomorphic function of the variables $w_1, \cdots, w_{n-1}, z$ in the domain under consideration (by Theorem 3.12). Thus the function $I_{\beta_2\beta_1}$ turns out to be an analytic (and multi-valued) function of the variables $w_1, \cdots, w_{n-1}, z$ in $Q \times \overline{p}_\rho$, except at the points $z = z_1$, $z = z_2$. We agree to choose, in (5.14), those branches of $\ln(z_2 - z)$ and $\ln(z_1 - z)$ for which the line of the cross-cut is the interval $l_{\beta_1\beta_2}$. Then for $\ln((z_2 - z)/(z_1 - z))$ we must take $\ln 1 = 0$. Thus, the choice of a branch of $\ln(z_1 - z)$ prescribes the choice of the branch of $\ln(z_2 - z)$. With respect to the functions $I_{\beta_2\beta_1}$ corresponding to various segments $l_{\beta_1\beta_2}$ converging at some one vertex Z of our network of squares, we choose branches of $\ln(Z - z)$ (in this connection see the remark made during the derivation of formula (5.19)).

With respect to the variable z the integral (5.9) is of the Cauchy type and therefore always defines a holomorphic function of z for all points of the plane of this variable that do not belong to the domain p_ρ. Therefore, for these values of z and for $(w_1, \cdots, w_{n-1}) \in Q$ the integral $I_{\beta_2\beta_1}$ is a holomorphic function of the variables $w_1, \cdots, w_{n-1}, z$ (but, of course, the representation (5.14) does not serve in this case).

Suppose that $P(w_1, \cdots, w_{n-1}, z)$ is a point of $Q \times l_{\beta_1\beta_2}$ $(z \neq z_1, z_2)$. We denote by $I^+_{\beta_2\beta_1}$ the limiting value of the function $I_{\beta_2\beta_1}$ when the point $(w_1, \cdots, w_{n-1}, \zeta)$ approaches the point $P \in Q \times q_{\beta_1}$, and by $I^-_{\beta_2\beta_1}$ the limiting value of the function $I_{\beta_2\beta_1}$ when the point $(w_1, \cdots, w_{n-1}, \zeta)$ approaches the point $P \in Q \times q_{\beta_2}$.

It is clear that (5.14) then implies

$$I^+_{\beta_2\beta_1} - I^-_{\beta_2\beta_1} = f_{\beta_2} - f_{\beta_1} \tag{5.15}$$

or

$$I^+_{\beta_2\beta_1} + f_{\beta_1} = I^-_{\beta_2\beta_1} + f_{\beta_2}. \tag{5.16}$$

We remark that from (5.9) we also obtain

$$I_{\beta_2\beta_1} = I_{\beta_1\beta_2}. \tag{5.17}$$

Now we consider the sum

$$\Phi(w_1, \cdots, w_{n-1}, z) = \sum_{(\beta_2 \beta_1)} I_{\beta_2 \beta_1}(w_1, \cdots, w_{n-1}, z), \tag{5.18}$$

taken over all pairs of mutually adjoining squares q_{β_1}, q_{β_2} (every pair is taken once). The value of this sum is also computed at the points corresponding to vertices of our network of squares. If Z is such a vertex, and the squares meeting at it correspond to the indices $\beta = 1, 2, 3, 4$, then in (5.18) there will be four terms I_{12}, I_{23}, I_{34}, I_{41} having the points $P^*(w_1, \cdots, w_{n-1}, z)$ (where $(w_1, \cdots, w_{n-1}) \in Q$) as singular points.

The cross-cuts for these functions are the sides which meet at the point Z and separate the squares q_1, q_2, q_3, and q_4 from one another. The corresponding branches of the function $\ln(Z - z)$ are to be chosen as follows: in the expression $I_{\gamma_1 \gamma_2}$ we must choose for q_β the value $\ln(Z - z) = \ln|Z - z| + i \arg(Z - z) + 2\pi m^{\beta}_{\gamma_1 \gamma_2} i$. Here $m^{\beta}_{\gamma_1 \gamma_2}$ is some suitably chosen integer.[1)]

Then at the points M lying in the neighborhood of the point P^* (the z-coordinates of the points M belong to q_β) the function Φ is represented in the following form

$$\left.\begin{aligned} \Phi = {} & \frac{f_2 - f_1}{2\pi i} \ln(Z - z) + \frac{f_3 - f_2}{2\pi i} \ln(Z - z) + \\ & + \frac{f_4 - f_3}{2\pi i} \ln(Z - z) + \frac{f_1 - f_4}{2\pi i} \ln(Z - z) + \\ & + m^{\beta}_{12}(f_2 - f_1) + m^{\beta}_{23}(f_3 - f_2) + m^{\beta}_{34}(f_4 - f_3) + \\ & + m^{\beta}_{41}(f_1 - f_4) + C = m^{\beta}_{12}(f_2 - f_1) + m^{\beta}_{23}(f_3 - f_2) + \\ & + m^{\beta}_{34}(f_4 - f_3) + m^{\beta}_{41}(f_1 - f_4) + C. \end{aligned}\right\} \tag{5.19}$$

Here $C = C(w_1, \cdots, w_{n-1}, z)$ is a holomorphic function of its variables in the relevant neighborhood, and the symbol $\ln(Z - z)$ always denotes that branch of the logarithm for which $\ln 1 = 0$.

In this way the function Φ, actually defined in the domain $Q \times q_\beta$, is extended to the points $(w_1, \cdots, w_{n-1}, z)$ corresponding to vertices of the squares q_β. We remark that at the points corresponding to vertices of the squares q_β which lie on the boundary of S_R, the function Φ has essential singularities.

Now we define in the domain $Q \times q_\beta$ the function

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1) If the side l_{12} has, for example, the direction of the imaginary axis, then q_1 lies in the second quadrant, q_2 in the first, q_3 in the fourth, and q_4 in the third. Then $m^{(1)}_{12} = m^{(4)}_{12} = m^{(3)}_{12} = 0$, $m^{(2)}_{12} = 1$.

$$g_\beta(w_1, \cdots, w_{n-1}, z) = \\ = f_\beta(w_1, \cdots, w_{n-1}, z) + \Phi_\beta(w_1, \cdots, w_{n-1}, z). \tag{5.20}$$

Here Φ_β is the value of the function Φ in the domain $Q \times q_\beta$. In view of (5.16) the functions g_β, so defined in the regions $Q \times q_\beta$, are extensions of one another. Thus, throughout the domain $Q \times S_R$ we have defined a (single-valued) meromorphic function which in each domain $Q \times S_R$ is equivalent with respect to subtraction to the previously prescribed function f_β. This function we will from now on denote by $g_{\alpha_1 \cdots \alpha_{n-1}}(w_1, \cdots, w_{n-1}, z)$.

We construct these functions $g_{\alpha_1 \cdots \alpha_{n-1}}$ for all domains $Q_{\alpha_1 \cdots \alpha_{n-1}} \times S_R$ ($Q_{\alpha_1 \cdots \alpha_{n-1}}$ here denotes the region which up to now we have labelled simply as Q). The functions g_β constructed for the domains $Q_{\alpha_1 \cdots \alpha_{n-1}} \times q_\beta$ are equivalent to the given functions f; therefore the functions $g_{\alpha_1 \cdots \alpha_{n-1}}$, defined in the adjacent domains $Q_{\alpha_1 \cdots \alpha_{n-1}} \times S_R$ will be equivalent at common points of these domains (we recall that the squares $Q_{\alpha_k}^{(k)}$ are somewhat larger than the squares $q_{\alpha_k}^{(k)}$, and therefore the domains $Q_{\alpha_1 \cdots \alpha_{n-1}} \times S_R$ have points in common).

We now alter the roles and notational symbols of our variables. We write $z = z_n$, and set $w_{n-1} = z_{n-1}$, and also, for S_R we write $S_R^{(n)}$. The role of the domain Q now belongs to the domain $Q_{\alpha_1}^{(1)} \times \cdots \times Q_{\alpha_{n-2}}^{(n-2)} \times S_R^{(n)}$. The square $q_{\alpha_{n-1}}^{(n-1)}$ will be called q_β. Repeating our reasoning, we define in $Q_{\alpha_1}^{(1)} \times \cdots \times Q_{\alpha_{n-2}}^{(n-2)} \times S_R^{(n-1)} \times S_R^n$ the meromorphic functions $g_{\alpha_1 \cdots \alpha_{n-2}}(w_1, \cdots, w_{n-2}, z_{n-1}, z_n)$ equivalent to the corresponding pre-assigned functions f. Next we "paste together" these functions along the w_{n-2}-plane (we rename w_{n-2}, putting $w_{n-2} = z_{n-2}$). In the end, we construct in the hypercube $S_R^{(1)} \times \cdots \times S_R^{(n)} = \Sigma_R$ a meromorphic function $\Psi_R(z_1, \cdots, z_n)$, equivalent at the points P of the domain Σ_R to the pre-assigned functions $f_p(z_1, \cdots, z_n)$. It remains for us to extend the definition of this function to the whole space.

First we assume that this function Ψ has been constructed for a hypercube of some radius $R' > R$, and therefore that the requirements of our theorem are fulfilled for $\Psi_R(z_1, \cdots, z_n)$ in the closed domain $\overline{\Sigma}_R$.

Next we consider the values $R = 1, 2, \cdots$ corresponding to the domain Σ_1, Σ_2, and the functions $\Psi_1(z_1, \cdots, z_n)$, $\Psi_2(z_1, \cdots, z_n)$, $\cdots$. The function

$$\Omega_m = \Psi_{m+1} - \Psi_m \tag{5.21}$$

is holomorphic in the closed domain $\overline{\Sigma}_m$. We expand Ω_m in a Taylor series with center at the origin of coordinates. This series will converge uniformly in the

closed polycylinder $\overline{B}_m \{ |z_k| \leq m;\ k = 1, \cdots, m\}$. Let

$$\epsilon_1 + \epsilon_2 + \cdots + \epsilon_k + \cdots \tag{5.22}$$

be a convergent series; with all $\epsilon_k > 0$. Then we split the Taylor series for Ω_m into two parts such that

$$\Omega_m = P_m + R_m,\ |R_m| < \epsilon_m$$

$$(\text{for } |z_k| \leq m;\ k = 1, \cdots, n), \tag{5.23}$$

where P_m is some polynomial. Now we choose instead of the function Ψ_m the functions

$$F_1 = \Psi_1,\ F_m = \Psi_m - \sum_{k=1}^{m-1} P_k \quad (\text{for } m \geq 2). \tag{5.24}$$

It is evident that the function F_m is equivalent to the function Ψ_m. Moreover, in the closed domain $\overline{B}_m$

$$F_{m+1} - F_m = R_m. \tag{5.25}$$

Finally, we form in the closed domain $\overline{B}_m$ the function

$$F = F_m + \sum_{k=m}^{\infty} R_k. \tag{5.26}$$

This function: 1) is equivalent in the closed domain $\overline{B}_m$ to the functions F_m and Ψ_m, since in view of (5.23) the second term on the right-hand side of equation (5.26) is represented in the closed domain $\overline{B}_m$ as the sum of a uniformly convergent series of holomorphic functions; 2) in the closed domains $\overline{B}_\nu (\nu < m)$ coincides with the function defined for the closed domain $\overline{B}_\nu$ by analogy with (5.26). In this latter case it is easy to verify equations (5.25) and (5.26).

The number m is arbitrary, and by allowing it to increase without limit we extend the function $F(z_1, \cdots, z_n)$ to the whole space C^n. This function satisfies all conditions of our theorem.

Now we turn to the case of an arbitrary polycylindrical domain $T = T_1 \times \cdots \cdots \times T_n$ of the space C^n of the variables $z_1, \cdots, z_n$.

For the proof we approximate to each domain T_k from within by a principal sequence[1] of domains $T_k^{(k)}$ consisting of squares. The remainder of the

1) A sequence of domains D_p, $p = 1, 2, \cdots$, satisfying the conditions that 1) $D_p \subset D_{p+1} \subset D$ for all indices p and 2) for every point $M \in D$ there exists a number p_M such that $U_M \subset D_p$ for $p > p_M$, is said to be a principal sequence of domains approximating the domain D from within. Here U_M is some suitably chosen neighborhood of the point M.

argument is the same as that given above for the open space. It must be emphasized that we are speaking only of single-valued functions.

2. Cousin's second theorem for the space C^n. This theorem is concerned with generalizing the Weierstrass theorem on the construction of an entire function with prescribed zeros.

We agree to say that two meromorphic functions, and in particular two holomorphic functions, are equivalent at the point P with respect to division if their quotient is holomorphic and different from zero at this point.

THEOREM 25.2 (second theorem of Cousin). *Suppose that to each point P of the space C^n there correspond some neighborhood V_P of P and a function f_P holomorphic in that neighborhood. Suppose also that if the neighborhoods V_P and V_Q of the points P, $Q \in C^n$ have points in common, the functions f_P and f_Q are equivalent with respect to division at those points. Then there exists an entire function F equivalent with respect to division at every point $P \in C^n$ to the given function f_P.*

PROOF. The proof of this theorem is wholly analogous to the proof of Cousin's first theorem. We use the same notation for domains and variables and merely indicate the necessary modifications of the reasoning. Thus, instead of the function $\phi(z)$ we must now deal with the function

$$g(w_1, \cdots, w_{n-1}, z) = \ln \frac{f_{\beta_2}(w_1, \cdots, w_{n-1}, z)}{f_{\beta_1}(w_1, \cdots, w_{n-1}, z)}.$$

Since the ratio f_{β_2}/f_{β_1} is holomorphic and non-vanishing in the neighborhood of the interval $l_{\beta_1\beta_2}$, the function g is holomorphic in that neighborhood. Correspondingly, in place of (5.9) we choose the integral

$$I_{\beta_2\beta_1}(w_1, \cdots, w_{n-1}, z) = \frac{1}{2\pi i} \int_{l_{\beta_1\beta_2}} \frac{g(w_1, \cdots, w_{n-1}, t)}{t - z}\, dt. \tag{5.27}$$

Then, as before, we form the sum $\Sigma I_{\beta_2\beta_1} = \Phi_1$ over all pairs $(\beta_2\beta_1)$. We consider the function Φ_1 in the neighborhood of some point which projects onto an inner vertex z of our network of squares. Repeating the argument leading to formula (5.19), we obtain an analogous representation of the function Φ_1 in the neighborhood of that point.

The coefficient of $\ln(Z - z)$ in this representation will have the form

$$\frac{1}{2\pi i}\left[\ln\frac{f_2}{f_1}+\ln\frac{f_3}{f_2}+\ln\frac{f_4}{f_3}+\ln\frac{f_1}{f_4}\right]=\frac{1}{2\pi i}\ln 1=\nu_z. \tag{5.28}$$

Here ν_z is some integer defined by the choice of the branch of the logarithm in the expression for the function g. We also form the function

$$\Phi_2=-\Sigma\nu_z\ln(Z-z). \tag{5.29}$$

Here the sum is taken over all inner vertices Z of our network.

Then the function $\Phi=\Phi_1+\Phi_2$ is extended to the boundary points of $Q\times q_\beta$ corresponding to the vertices of q_β. Now, instead of (5.20), we take in each domain $Q\times q_\beta$ the function

$$G_\beta(w_1,\cdots,w_{n-1},z)=f_\beta e^{\Phi_\beta}. \tag{5.30}$$

Here Φ_β is the value of the function Φ at points on the boundary of $Q\times q_\beta$. After this, just as in the proof of Cousin's first theorem, we use the "pasting" process to construct a function $\Psi_R(z_1,\cdots,z_n)$ satisfying the requirements of our theorem in the closed domain $\overline{\Sigma}_R=\overline{S}_R^{(1)}\times\cdots\times\overline{S}_R^{(n)}$ (to obtain this function in $\overline{\Sigma}_R$ we carry out the reasoning for some $R'>R$).

Setting $R=1,2,\cdots$, we obtain a sequence of functions $\Psi_1,\Psi_2,\cdots$. By the nature of its definition the function

$$\frac{\Psi_{m+1}(z_1,\cdots,z_n)}{\Psi_m(z_1,\cdots,z_n)}=e^{\Omega_m(z_1,\cdots,z_n)} \tag{5.31}$$

is holomorphic in the closed domain $\overline{\Sigma}_m$ and nowhere vanishes there. Therefore the function $\Omega_m(z_1,\cdots,z_n)$ must be holomorphic in the closed domain Σ_m.

Just as in the proof of Cousin's first theorem, we define for $\Omega_m(z_1,\cdots,z_n)$ the functions P_m and R_m and use them to construct the function

$$F_1=\Psi_1;\ F_s=\Psi_s e^{-\Sigma_{k=1}^{s-1}P_k}\quad(s>1). \tag{5.32}$$

Finally, we define in the closed polycylinder $\overline{B}_m\{|z_k|\le m,\ k=1,\cdots,n\}$ the function

$$F(z_1,\cdots,z_n)=F_m e^{\Sigma_{k=m}^{\infty}R_k} \tag{5.33}$$

By its construction, the function F has the following properties:

1) in the closed polycylinder $\overline{B}_m$ it is equivalent to the given functions f_p;

2) in the domain B_μ $(\mu\le m)$ it coincides with the functions constructed in the

closed polycylinder $\overline{B}_\mu$ by analogy with (5.33).

Here m is an arbitrary number; therefore the function F is defined in the whole space C^n and satisfies there all requirements of Cousin's second theorem.

3. **Cousin's second theorem for polycylindrical domains.** Cousin's second theorem can be generalized, in the same way as the first, to the case of polycylindrical domains. If, however, the domain Q consists of the Cartesian product of regions not in general simply connected, then $\ln (f_{\beta_2}/f_{\beta_1})$ will perhaps not have single-valued analytic, i.e., holomorphic, branches in the domain $Q \times P_\rho$ (here P_ρ denotes, as in the proof of Cousin's first theorem, the relevant neighborhood of the interval $l_{\beta_1 \beta_2}$ in the z-plane). In this case there may exist no single-valued analytic, i.e., holomorphic, function satisfying the conditions of Cousin's second theorem in such a polycylindrical domain. Therefore in order to arrive at a holomorphic function satisfying the requirements of this theorem in the polycylindrical domain $T_1 \times \cdots \times T_n$, we must demand that all the plane regions T_k be simply connected, with the possible exception of one (the domain Q is defined as the product of domains simultaneously approximating these simply connected regions; multiple connectivity of a domain T_k which does not take part in the formation of Q is of no relevance). We arrive at the following theorem:

THEOREM 25.3. *Suppose that to every point P of some polycylindrical domain $T = T_1 \times \cdots \times T_n$ (where all these domains T_k, with the possible exception of one, are simply connected) there correspond a neighborhood V_P of P and a function f_P holomorphic in it. Suppose further that if any two neighborhoods V_P and V_Q have points in common, the functions f_P and f_Q are equivalent with respect to division at those points. Then in the domain T there exists a holomorphic function equivalent to the prescribed function at each point P of T.*

In this form the theorem was first announced by Gronwall [1]. Cousin himself did not notice that a function satisfying the conditions of this theorem in a polycylindrical domain for which two (or more) of the domains T_k are multiply-connected is many valued.

Gronwall, in the paper cited above, constructed an example of a bicylindrical domain $T_1 \times T_2$ in the space of the variables w, z with multiply-connected domains T_1 and T_2 for which it is impossible to construct a holomorphic function satisfying the requirements of Cousin's second problem. Then a number of other mathematicians constructed a further series of (simpler) examples. We now present an example developed by K. Oka [1] which is important because it indicates

a way to study the possibility of further generalizations of Cousin's second theorem.

Suppose that in the neighborhoods V_P and V_Q of the points $P, Q \in C^n$ there are given continuous, and, in particular, holomorphic, functions f_P and f_Q. We shall say that these are equivalent in the generalized sense if in the common part $V_P \cap V_Q$ of these neighborhoods the ratio f_P/f_Q is continuous and different from zero. It is quite obvious that if in some domain D for some system of holomorphic functions $\{f_P,\ P \in D\}$ it is impossible to construct a continuous function F equivalent to these functions in the generalized sense throughout the domain D, then there exists no holomorphic function equivalent in the domain D to the functions $\{f_P\}$ in the ordinary sense.

Let $T_1 \times T_2$ be a bicylindrical domain in the space C^2 of the variables w, z with $T_1\{r_1 < |w| < 1\}$, $T_2\{r_2 < |z| < 1\}$; $r_1 + r_2 > 1$. We take the analytic plane $w - z - 1 = 0$. It is evident at once that two disjoint pieces of it lie in the domain $T_1 \times T_2$. For the points of one piece $w + \overline{w} > 0$, $z + \overline{z} > 0$, and for points of the other $w + \overline{w} < 0$, $z + \overline{z} < 0$; we denote the first piece by σ.

Let us suppose that in the domain $T_1 \times T_2$ there exists a continuous function $F(w, z)$ that is equivalent in the generalized sense to the function $w - z - 1$ in the neighborhood $\overline{\sigma}$ of the piece σ and to unity on the remaining points (the neighborhoods of these latter points being so chosen that they do not contain points of the piece σ).

We select further the circle $C\{|z| = \rho,\ r_2 < \rho < 1\}$ and a point $w_0 \in T_0$ such that the function $F(w_0, z)$ does not vanish on the circle C. We denote by $\delta(F, w_0)$ the increment in the argument of $F(w_0, z)$ corresponding to a single traversal of the circle C in the positive direction. Now we consider the quantities $w = \omega_1$, $w = \omega_2$, where $r_1 < \omega_1 < 1$; $-1 < \omega_2 < -r_1$. Then the function $F(w, z)$ does not vanish on the surface $\gamma \times T_2$ where γ is a curve joining ω_1 and ω_2, including these points and lying entirely in the lower half of the annulus T_1. The function $\delta(F, w)$ is defined and continuous at all points of γ. Since it can take on only a discrete set of values, $\delta(F, \omega_1) = \delta(F, \omega_2)$.

On the other hand, suppose that $T_1^* = T_1 \cap \{\operatorname{Im} w \geq 0\}$. In a suitably chosen neighborhood of the points $(w, z) \in T_1^* \times T_2$ the functions $F(w, z)$ and $w - z - 1$ are equivalent to one another in the generalized sense, and therefore there exists

in this neighborhood a continuous non-vanishing function $\gamma(w, z)$[1] such that

$$F(w, z) = (w - z - 1)\,\lambda(w, z). \tag{5.34}$$

The function $\lambda(w, z) \neq 0$ for $(w, z) \in T_1^* \times T_2$. Arguing in this case just as we did for the function $F(w, z)$ (except that we join the points ω_1 and ω_2 by a curve lying in the upper half of the annulus) we find that $\delta(\lambda, \omega_2) = \delta(\lambda, \omega_1)$. Then $\delta(w - z - 1, \omega_1) = \delta(\omega_1 - z - 1)$ must, in view of (5.34), be equal to $\delta(w - z - 1, \omega_2) = \delta(\omega_2 - z - 1)$. But it is immediately evident that $\delta(\omega_1 - z - 1) = -2\pi$, $\delta(\omega_2 - z - 1) = 0$.

Thus we have arrived at a contradiction, so that in the present case no function exists in the domain $T_1 \times T_2$ which is equivalent with respect to division, even in the generalized sense, to the prescribed functions f_p.

§ 26. CHARACTERISTICS OF THE GROWTH OF AN ENTIRE FUNCTION

The quantities that we consider below either characterize the growth of an entire function $f(z)$, where $z \in C^n$, throughout the entire space C^n, or characterize its behavior on some specially chosen surface in the space C^n.

1. Order and type are the most commonly encountered characteristics of an entire function $f(z)$, where $z \in C^1$. For such functions the order ρ and the type σ are defined as the lower bounds of the numbers ν and μ, respectively, for which we have the asymptotic[2] inequalities

$$M_f(R) = e^{R^\nu};\; M_f(R) < e^{\mu R^\rho}, \tag{5.35}$$

where

$$M_f(R) = \sup_{\left|\frac{z}{R}\right| < 1} |f(z)|. \tag{5.36}$$

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1) We note that the fact that two continuous functions F_1 and F_2 are equivalent with respect to division (in the generalized sense) at all points of some domain D (or, as one says, are "locally equivalent") does not in general imply the existence of a continuous non-vanishing function λ such that $F_1 = \lambda F_2$. Nevertheless, if the set of points $D \cap \{F_1 = F_2 = 0\}$ has no inner points, the existence of such a function is obvious. In this case all the points of D at which F_1/F_2 is not defined are limit points of those points of the domain D at which the function $\lambda = F_1/F_2$ is defined and continuous. The existence of a limit for the function F_1/F_2 everywhere in D is specified by the local equivalence condition. Defining the function λ at the points of the set $D \cap \{F_1 = F_2 = 0\}$ by a limiting process, we obtain it as a continuous non-vanishing function throughout the domain D. Thus, in this case the functions F_1 and F_2 are equivalent in the large in the domain D (with respect to division in the generalized sense).

2) An inequality containing a variable quantity is said to be asymptotic if it is satisfied for all sufficiently large values of that variable.

These quantities ρ and σ are calculated by means of formulas which also serve to define them:

$$\rho = \overline{\lim_{R\to\infty}} \frac{\ln \ln M_f(R)}{\ln R}; \quad \sigma = \overline{\lim_{R\to\infty}} \frac{\ln M_f(R)}{R^\rho}. \tag{5.37}$$

In the general case of the space C^n, where $n \geq 1$, we follow A. A. Gol'dberg [1] and instead of the disk $|z| < 1$ take an arbitrary bounded complete n-circular domain $D \subset C^n$ with center at the origin of coordinates. Then we write for the entire function

$$f(z) = \sum_k c_k z^k, \tag{5.38}$$

$$M_{f,D}(R) = \sup_{z \in D_R} |f(z)|,$$

$$\rho_D = \overline{\lim_{R\to\infty}} \frac{\ln \ln M_{f,D}(R)}{\ln R}; \quad \sigma_D = \overline{\lim_{R\to\infty}} \frac{\ln M_{f,D}(R)}{R^{\rho_D}}. \tag{5.37$_1$}$$

Here, as usual, k is a vector with components $k_1, \cdots, k_n$ which take on the values $0, 1, 2, \cdots$; and the point $z \in D_R$ if the point $(z_1/R, \cdots, z_n/R) \in D$.

The following theorem establishes a connection between these quantities and the coefficients of the expansion (5.38). It turns out that the quantity ρ_D does not depend on the choice of the domain D. The numbers $\rho = \rho_D$ and σ_D are respectively called the *order* and the *D-type* of the function $F(z)$.

THEOREM 26.1 (Gol'dberg [1]). *The equations*

$$\rho_D = \rho = \overline{\lim_{\|k\|\to\infty}} \frac{\|k\| \ln \|k\|}{-\ln \|c_k\|}, \tag{5.39}$$

$$(e\rho\sigma_D)^{\frac{1}{\rho}} = \overline{\lim_{\|k\|\to\infty}} \{\|k\|^{\frac{1}{\rho}} [|c_k| d_k(D)]^{\frac{1}{\|k\|}}\}^{1)} \tag{5.40}$$

are valid. Here, as before, $\|k\| = k_1 + \cdots + k_n$, $d_k(D) = \sup_{z\in D} |z|^k$, *where* $|z|^k = |z_1|^{k_1} \cdots |z_n|^{k_n}$.

PROOF.[2] We consider a function of the complex variables $u, z_1, \cdots, z_n$:

1) For particular forms of the region D formulas (5.39) and (5.40) were obtained earlier in papers by J. Sire [1], A. A. Temljakov [1], and S. A. Eremin [1].

2) The proof is carried out in essentially the same way as for the one-variable case. See, for example, B. Ja. Levin, *Distribution of zeros of entire functions*, GITTL, Moscow, 1956 (Russian); English transl., Amer. Math. Soc., Providence, R. I., 1963. In what follows we shall cite this book as: Levin, *Entire functions*.

$$f(u, z) = \sum_{\kappa=0}^{\infty} u^{\kappa} p_{\kappa}(z), \tag{5.41}$$

where

$$p_{\kappa}(z) = \sum_{\|k\|=\kappa} c_k z^k. \tag{5.42}$$

For $u = 1$ the function $f(u, z)$ coincides with the entire function $f(z)$ defined by the series (5.38). Considering the function $f(u, z)$ as a function of the variable u alone, and applying Cauchy's inequality, we obtain

$$M_{p_{\kappa}, D}(1) \leq \frac{M_{f, D}(R)}{R^{\kappa}}. \tag{5.43}$$

It follows that if the quantity $M_{f, D}(R)$ satisfies the asymptotic inequality (asymptotic with respect to the value of the quantity R)

$$M_{f, D}(R) \leq e^{\mu R^{\nu}}, \tag{5.44}$$

then

$$M_{p_{\kappa}, D}(1) \leq e^{\mu R^{\nu}} R^{-\kappa}.$$

Minimizing the right-hand side of this inequality, we find that asymptotically (with respect to κ)

$$M_{p_{\kappa}, D}(1) \leq \left[\frac{e\nu\mu}{\kappa}\right]^{\kappa/\nu}. \tag{5.45}$$

Suppose that the point $z' \in \partial D$ is so chosen that $|z'|^k = d_k(D)$. Then, since D is a complete n-circular domain, the polycylinder $E_k\{|z_j| < |z'_j|, j = 1, \cdots, n\} \subset D$. Therefore

$$M_{p_{\kappa}, E_k}(1) \leq M_{p_{\kappa}, D}(1).$$

From this, and from Cauchy's inequality it follows that

$$|c_k| \leq \frac{M_{p_{\kappa}, D}(1)}{d_k(D)} \quad (\text{where } \|k\| = \kappa)$$

and accordingly

$$|c_k| d_k(D) \leq \left[\frac{e\mu\nu}{\kappa}\right]^{\kappa/\nu} (\text{where } \|k\| = \kappa). \tag{5.46}$$

We now postulate that from some $\kappa = \kappa_0$ onward the inequality (5.46) is satisfied. Then

$$M_{f, D}(R) \leq \sum_{\kappa=0}^{\infty} \left[R^{\kappa} \sum_{\|k\|=\kappa} |c_k| d_k(D)\right] \leq$$

$$\leq \sum_{\kappa=0}^{\infty} R^{\kappa}(\kappa+1)^n \left[\frac{e\mu\nu}{\kappa}\right]^{\kappa/\nu} + C(1+R)^{\kappa_0}, \qquad (5.47)$$

where C is a suitably chosen constant. We write $N(R) = 2^{\nu} e\mu\nu R^{\nu}$ and split the last sum in (5.47) into two: for $\kappa \leq N(R)$ and for $\kappa > N(R)$. Evidently, every term of the second sum is less than $2^{-\kappa}\kappa^n$, and

$$\sup_{\kappa}\left[\left[\frac{e\mu\nu}{\kappa}\right]^{\kappa/\nu} R^{\kappa}\right] = e^{\mu R^{\nu}}.$$

Therefore

$$M_{f,D}(R) \leq C(1+R)^{\kappa_0} + \sum_{\kappa\leq N(R)} (1+\kappa)^n e^{\mu R^{\nu}} +$$

$$+ \sum_{\kappa>N(R)} (1+\kappa)^n 2^{-\kappa} \leq C(1+R)^{\kappa_0} + C_1 + (1+N(R))^{n+1} e^{\mu R^{\nu}},$$

where

$$C_1 = \sum_{k=1}^{\infty} \frac{k^n}{2^k}.$$

From the estimate we have obtained it follows (since $N(R)$ increases like R^{ν}) that for arbitrary $\nu_1 > \nu$ we have asymptotically

$$M_{f,D}(R) \leq e^{\mu R^{\nu_1}}. \qquad (5.48)$$

Thus, from the inequality (5.44) we obtain the inequality (5.46); on the other hand, the inequality (5.46) implies the inequality (5.48). In the latter we have in place of ν the arbitrary quantity $\nu_1 > \nu$; however, the lower bounds of the numbers ν and μ for which the inequalities (5.44) and (5.46) are fulfilled evidently coincide.

Hence by an immediate calculation we find that

$$\rho_D = \overline{\lim_{\|k\|\to\infty}} \frac{\|k\|\ln\|k\|}{-\ln\{|c_k| d_k(D)\}},$$

and that for the quantities σ_D we obtain formula (5.40) with $\rho = \rho_D$.

It remains for us to show that the quantity ρ_D does not depend on the choice of the domain D.

We choose the polycylinder $E_q\{|z_j| < A^{(q)},\ j = 1, \cdots, n,\ q = 1, 2\}$ so that $E_1 \subset D \subset E_2$. Then

$$M_{f,E_1}(R) \leq M_{f,D}(R) \leq M_{f,E_2}(R).$$

It is easy to see that $\rho_{E_1} = \rho_{E_2}$ for arbitrary $A^{(1)}$ and $A^{(2)}$. From this our assertion about ρ_D follows. Now to obtain formula (5.39) it is sufficient to choose $D\{|z_j| < 1,\ j = 1, \cdots, n\}$.

REMARK. In applying formula (5.40) it is useful to keep in mind the fact that for $D\{|z_1|^2 + \cdots + |z_n|^2 < 1\}$ the quantity $d_k(D) = \sqrt{k_1^{k_1} \cdots k_n^{k_n}\{\|k\|\}^{-\|k\|}}$, and for $D\{|z_1| + \cdots + |z_n| < 1\}$ the quantity $d_k(D) = k_1^{k_1} \cdots k_n^{k_n}\{\|k\|\}^{-\|k\|}$.

2. Hypersurfaces of conjugate orders and conjugate types.[1] For the entire function (5.38) we consider the quantity $M_f(R) = M_{f, E_R}(1)$, where $E_R\{|z_j| < R_j,\ j = 1, \cdots, n\}$. We shall compare the growth of the function $\ln M_f(R)$ with the growth of the function $R_1^{a_1} + \cdots + R_n^{a_n}$.

Suppose that $B_\rho \subset R_n$, where R_n is the space of the real numbers $a_1, \cdots, a_n$, is the set of points $a \in R_n$ for which asymptotically

$$\ln M_f(R) < R_1^{a_1} + \cdots + R_n^{a_n}. \tag{5.49}$$

Clearly, if the point $(a_1', \cdots, a_n') \in B_\rho$, then the whole octant $\{a_j \geq a_j',\ j = 1, \cdots, n\} \subset B_\rho$. On the other hand, if the point $(a_1', \cdots, a_n') \overline{\in} B_\rho$, then all points of the region $\{a_j < a_j',\ j = 1, \cdots, n\}$ do not belong to the set B_ρ. The boundary $\partial B_\rho = S_\rho$ separates the whole space R_n into two parts: in one part the inequality (5.49) is satisfied, and in the other it is not. The boundary S_ρ of the set B_ρ is called the *hypersurface of conjugate orders of the function* $f(z)$. A system of numbers $\rho_1, \cdots, \rho_n$ is called a *system of conjugate orders* of the function $f(z)$ if $(\rho_1, \cdots, \rho_n) \in S_\rho$. It can be shown that the positive numbers $\rho_1, \cdots, \rho_n$ form a system of conjugate orders if and only if

$$\varlimsup_{\|R\| \to \infty} \frac{\ln \ln M_f(R)}{\ln(R_1^{\rho_1} + \cdots + R_n^{\rho_n})} = 1.$$

Suppose that $\rho_1, \cdots, \rho_n$ is a system of conjugate orders of the function $f(z)$. To obtain a more complete characterization of the growth of this function we compare it with the growth of the function $a_1 R_1^{\rho_1} + \cdots + a_n R_n^{\rho_n} = \|aR^\rho\|$. We denote by B_σ the set of all points $a \in R_n$ for which, asymptotically,

$$\ln M_f(R) \leq \|aR^\rho\|. \tag{5.50}$$

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1) The concept of the hypersurface of conjugate types was first introduced by Ronkin [1, 3]. The method of definition adopted here for these hypersurfaces is essentially the same as that used in the papers cited.

Following established tradition we call this set a domain.[1] We show that the domain B_σ is convex.

In fact, suppose that the points $a', a'' \in B_\sigma$. Then in view of the definition of the set B_σ we have asymptotically

$$\ln M_f(R) \leq \|a'R^\rho\|, \quad \ln M_f(R) \leq \|a''R^\rho\|.$$

Accordingly, asymptotically

$$\ln M_f(R) = (1-t)\ln M_f(R) + t\ln M_f(R) \leq$$

$$\leq (1-t)\|a'R^\rho\| + t\|a''R^\rho\| \leq \sum_{j=1}^{n} [(1-t)\,a_j' + ta_j''] R_j^{\rho_j}.$$

Here $0 \leq t \leq 1$. Our assertion follows from this derived inequality.

The boundary $\partial B_\sigma = S_\sigma$ of the set B_σ is called the *hypersurface of conjugate types of order* $(\rho_1, \cdots, \rho_n)$ *of the function* $f(z)$. A system of numbers $\sigma_1, \cdots, \sigma_n$ is called a *system of conjugate types* of order $\rho_1, \cdots, \rho_n$ of the function $f(z)$ if $(\sigma_1, \cdots, \sigma_n) \in S_\sigma$.

It can be shown that the positive numbers $\sigma_1, \cdots, \sigma_n$ form a system of conjugate types of order $\rho_1, \cdots, \rho_n$ if and only if

$$\overset{*}{\overline{\lim_{\|R\|\to\infty}}} \frac{\ln M_f(R)}{\|\sigma R^\rho\|} = 1.$$

We establish a connection between conjugate orders, conjugate types, and the coefficients of the expansion of the function $f(z)$ in the series (5.38). We have

THEOREM 26.2. 1) *The positive numbers $\rho_1, \cdots, \rho_n$ form a system of conjugate orders of an entire function $f(z)$ if and only if*

$$\overline{\lim_{\|k\|\to\infty}} \frac{\left\|\frac{k}{\rho}\right\| \ln\|k\|}{-\ln|c_k|} = 1. \tag{5.51}$$

2) *The positive numbers $\sigma_1, \cdots, \sigma_n$ form a system of conjugate types of order $(\rho_1, \cdots, \rho_n)$ of an entire function $f(z)$ if and only if*

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1) The use of this term for the set B_σ (as well as for other sets defined later and characterizing the growth of an entire function) does not, however, imply that it is a domain in the topological sense of the word.

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k| \left[\frac{k}{\sigma e \rho}\right]^{k/\rho}} = 1. \tag{5.52}$$

Here

$$\left\|\frac{k}{\rho}\right\| = \frac{k_1}{\rho_1} + \cdots + \frac{k_n}{\rho_n}; \quad \left[\frac{k}{\sigma e \rho}\right]^{k/\rho} = \left[\frac{k_1}{\sigma_1 e \rho_1}\right]^{k_1/\rho_1} \cdots \left[\frac{k_n}{\sigma_n e \rho_n}\right]^{k_n/\rho_n}.$$

The proof of this theorem is omitted; to a great extent it duplicates the proof of Theorem 26.1. In the particular case when the domains B_ρ and B_σ represent octants of the form $\{a_j > a_j^{(0)},\ j = 1, \cdots, n\}$, formulas (5.51) and (5.52) were first obtained by M. M. Džerbašjan;[1] in the general case formula (5.51) was first obtained by A. A. Gol'dberg[2] and formula (5.52) by L. I. Ronkin.

Theorem 26.2 allows us to find characteristic properties of the hypersurfaces of conjugate orders and conjugate types. We have

THEOREM 26.3. *Let S be a hypersurface lying in the absolute octant R_n^+ of the space R_n of the variables $a_1, \cdots, a_n$, and S^{-1} the hypersurface obtained from S by the substitution $b_j = a_j^{-1}$, $j = 1, \cdots, n$. Then S is a hypersurface of conjugate orders for some entire function $f(z)$ if and only if the hypersurface S^{-1}, together with the coordinate hyperplanes, constitutes the boundary of some convex complete region D in the octant R_n^+.*

REMARK. We call the region D complete if together with every point $(b_1^{(0)}, \cdots, b_n^{(0)})$ contained in it, it also contains all points $(b_1, \cdots, b_n)$ for which $0 < b_j \le b_j^{(0)}$, $j = 1, \cdots, n$.

THEOREM 26.4. *Let S be a hypersurface lying in the absolute octant R_n^+ of the space R_n of the variables $a_1, \cdots, a_n$, and let S_1 be the hypersurface obtained from S by the substitution $b_j = \ln a_j$, $j = 1, \cdots, n$. Then the hypersurface S is a hypersurface of conjugate types of order $(\rho_1, \cdots, \rho_n)$ for some entire function $f(z)$ if and only if the hypersurface S_1 consists of the boundary of some convex octant-shaped region D.*

We shall prove only Theorem 26.3. Suppose that the points $(\rho_1^{(1)}, \cdots, \rho_n^{(1)})$, $(\rho_1^{(2)}, \cdots, \rho_n^{(2)}) \in B_\rho$. Then

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1) See Džerbašjan [1]. In this paper the expression (5.51) has a somewhat different form, but is nevertheless essentially equivalent to ours.

2) See Gol'dberg [1]. In Gol'dberg's paper [2] there are generalizations of Theorems 26.1 and 26.2.

$$\overline{\lim_{\|k\|\to\infty}} \frac{\left\|\frac{k}{\rho^{(s)}}\right\| \ln\|k\|}{-\ln|c_k|} \leq 1, \quad s = 1, 2.$$

We multiply the inequality for $s = 1$ by the number ν, where $0 < \nu < 1$, and the inequality for $s = 2$ by the number $\mu = 1 - \nu$. Adding these inequalities we obtain

$$\overline{\lim_{\|k\|\to\infty}} \frac{\left\|k\left[\frac{\nu}{\rho^{(1)}} + \frac{\mu}{\rho^{(2)}}\right]\right\| \ln\|k\|}{-\ln|c_k|} \leq 1, \tag{5.53}$$

where

$$\left\|k\left[\frac{\nu}{\rho^{(1)}} + \frac{\mu}{\rho^{(2)}}\right]\right\| = k_1\left[\frac{\nu}{\rho_1^{(1)}} + \frac{\mu}{\rho_1^{(2)}}\right] + \cdots + k_n\left[\frac{\nu}{\rho_n^{(1)}} + \frac{\mu}{\rho_n^{(2)}}\right].$$

It is now easy to see: if the points $P_1(b_1^{(1)}, \cdots, b_n^{(1)})$, $P_2(b_1^{(2)}, \cdots, b_n^{(2)}) \in B_\rho^{-1}$, then the points $(1/b_1^{(1)}, \cdots, 1/b_n^{(1)})$, $(1/b_1^{(2)}, \cdots, 1/b_n^{(2)}) \in B_\rho$; therefore in view of the inequality (5.53) an arbitrary point $(1/(\nu b_1^{(1)} + \mu b_1^{(2)}), \cdots$ $\cdots, 1/(\nu b_n^{(1)} + \mu b_n^{(2)})) \in B_\rho$, where $\nu, \mu > 0$, $\mu + \nu = 1$, and accordingly the whole interval P_1P_2 belongs to the region B_ρ^{-1}. Thus the region B_ρ^{-1} is convex; its completeness is immediately evident.

Thus, the necessity of the condition stated in Theorem 26.3 has been proved; let us turn to the proof of its sufficiency. Suppose that the hypersurface Σ together with the coordinate hyperplanes form the boundary of some convex complete region D lying in the absolute octant R_n^+ of the space R_n of the variables $b_1, \cdots, b_n$. We consider the support function of the region D

$$h_D(\alpha) = h_D(\alpha_1, \cdots, \alpha_n) = \sup_{(b_1,\cdots,b_n)\in \bar{D}} (b_1 \cos\alpha_1 + \cdots + b_n \cos\alpha_n).$$

Here $\alpha_1, \cdots, \alpha_n$ are the angles formed by the chosen ray with the coordinate axes. Since the region D is complete and lies in the octant R_n^+, the hypersurface Σ is completely defined by the values of the function $h_D(\alpha)$ for $\alpha_j \in [0, \pi/2]$ $(j = 1, \cdots, n)$.

Let us further consider the function

$$f(z) = \sum_k c_k z^k.$$

Here and later

$$c_{k_1,\cdots,k_n} = \exp\{-h_D(\alpha_1^{(k_1)},\cdots,\alpha_n^{(k_n)})\sqrt{\|k^2\|}\ln\|k\|\},$$

$$\alpha_j^{(k_j)} = \arccos\frac{k_j}{\sqrt{\|k^2\|}},\quad \|k^2\| = k_1^2+\cdots+k_n^2.$$

It is easy to see that the function $f(z)$ so defined is an entire function.

The definition of the support function of a region implies that for every point $b \in \Sigma$

$$\overline{\lim_{\|k\|\to\infty}}\frac{\|kb\|\ln\|k\|}{-\ln|c_k|} = \overline{\lim_{\|k\|\to\infty}}\frac{\|kb\|}{h_D(\alpha^{(k_1)},\cdots,\alpha^{(k_n)})\sqrt{\|k^2\|}} =$$

$$= \overline{\lim_{\|k\|\to\infty}}\frac{b_1\cos\alpha_1^{(k_1)}+\cdots+b_n\cos\alpha_n^{(k_n)}}{h_D(\alpha_1^{(k_1)},\cdots,\alpha_n^{(k_n)})} \le 1. \tag{5.54}$$

Here and later $\|kb\| = k_1b_1+\cdots+k_nb_n$.

Since $h_D(\alpha)$ is the support function of a convex region, for every point $b\in\Sigma$ there exists a direction $(\alpha_1,\cdots,\alpha_n)$ such that

$$b_1\cos\alpha_1+\cdots+b_n\cos\alpha_n = h_D(\alpha).$$

We form a sequence of vectors $k(k_1,\cdots,k_n)$ such that $\alpha_j^{(k_j)}\to\alpha_j$ for all $j=1,\cdots,n$. It is easy to see that this is always feasible.

From the continuity of the support function $h_D(\alpha)$ it follows that for the selected sequence vectors k

$$\lim_{\|k\|\to\infty}\frac{b_1\cos\alpha_1^{(k_1)}+\cdots+b_n\cos\alpha_n^{(k_n)}}{h_D(\alpha_1^{(k_1)},\cdots,\alpha_n^{(k_n)})} = 1.$$

From this, and from the inequality (5.54), we conclude that for the points $b\in\Sigma$

$$\overline{\lim_{\|k\|\to\infty}}\frac{\|kb\|\ln\|k\|}{-\ln|c_k|} = 1. \tag{5.55}$$

Taking equations (5.51) and (5.55) together, we find that the numbers $1/b_1,\cdots,1/b_n$ form a system of conjugate orders for the function $f(z)$. Therefore, the hypersurface S, which we obtained from the hypersurface Σ by the transformation $a_j = 1/b_j$, $j=1,\cdots,n$, is in fact a hypersurface of conjugate orders for the entire function $f(z)$ which we have constructed.

We shall not take time for the extension to the case of many variables of the

theory of Wiman-Valiron and the corresponding results of I. F. Bitljan, A. A. Gol'dberg [1], and Š. I. Strelic [1].

3. **Growth of an entire function with respect to one of its variables.** Suppose that $f(w, z_1, \cdots, z_n) = f(w, z)$ is an entire function of the variables $w, z_1, \cdots \cdots, z_n$. Fixing the variables $z_1, \cdots, z_n$ in one way or another, we consider the growth of the function f as a function of one variable w only. In the case $n = 1$ this growth has been quite fully studied by J. Sire [1], P. Lelong [1], and L. I. Ronkin [1]. We shall sketch the results so far obtained.

We denote by $\rho_f(z_1, \cdots, z_n)$ and $\sigma_f(z_1, \cdots, z_n)$ the order and type, respectively, of the function $f(w, z)$ with respect to the variable w. Then for $n = 1$ we have

THEOREM 26.5 (Lelong [1]). *If for some $R_2' > 0$ the function $M_f(R_1, R_2')$ has the order of growth ρ_1 with respect to the variable R_1, then for all values of the variable z,*

$$\rho_f(z) \leq \rho_1.$$

The strong inequality holds only on some set M_ρ of inner measure zero. Here $M_f(R_1, R_2) = \sup |f|$ for $|w| < R_1$, $|z| < R_2$.

THEOREM 26.6 (Ronkin [1]). *If for the entire function $f(w, z)$ and for arbitrary fixed R_2 we have asymptotically*

$$M_f(R_1, R_2) < e^{aR_1^{\rho_1}},$$

where a does not depend on R_2, and ρ_1 has the same meaning as in the preceding theorem, then

$$\sigma_f(z) = \sigma_{f,w}$$

everywhere with the possible exception of the points of a set M_σ representable in the form

$$M_\sigma = \bigcup_{j=1}^{\infty} A_j. \tag{5.56}$$

Here A_j is a closed set, of measure zero, of points of the z-plane and $A_j \subset A_{j+1}$ $(j = 1, 2, \cdots)$; $\sigma_{f,w} = \sup_{z \in C_z^1} \sigma_f(z)$.

The characterization of the sets M_ρ and M_σ given in these theorems is almost exact. We have

THEOREM 26.7 (Ronkin [1]). *If a certain set $M \subset C_z^1$ can be represented as*

a sum of the form (5.56), *then there exists an entire function* $f(w, z)$, *satisfying the conditions of Theorem* 26.5, *such that the corresponding set* M_σ *satisfies the inclusion relation*

$$M \subset M_\sigma \subset \overline{M}.$$

We can conclude from Theorems 26.5–26.7 that the growth of the function $f(w, z)$ is the same, generally speaking, for all values of z.[1)]

For entire functions of $n > 2$ variables there exist theorems which are similar in a general way to Theorems 26.5 and 26.6. However, the structure of the sets M_ρ and M_σ in this case has not yet been sufficiently clarified. It is known for the set M_σ that its intersection $M_\sigma \cap E$ with an arbitrary analytic plane $E\{z_j = a_j\tau + b_j,\ j = 1, \cdots, n\}$ not belonging entirely to it, where τ is a complex parameter, can be represented in the form of a sum (5.56). From this we may conclude that the spatial Lebesgue measure of the set M_σ is zero. Similar results hold for the set M_ρ as well. Just how precisely these properties characterize the sets M_ρ and M_σ remains an open question.

A weaker characterization (as compared to the type) of the growth of an entire function of order ρ in one variable z is afforded by its growth indicator $h(\theta)$. This is defined by the equation

$$h(\theta) = \overline{\lim_{r\to\infty}}\,(r^{-\rho}\ln|f(re^{i\theta})|).$$

For entire functions of two complex variables we have

THEOREM 26.8 (Ronkin [3]). *If an entire function* $f(w, z)$ *of order* $\rho = 1$ *satisfies the conditions of Theorem* 26.6, *then the set* N_w *of points* z *at which for at least one value of* θ *the inequality*

$$h(\theta, z) < h_w(\theta)$$

is satisfied can be represented in the form of a sum (5.56).

Here $h(\theta, z)$ and $h_w(\theta)$ are quantities defined (in the general case for $n \geq 1$) by the equations

$$h(\theta, z) = \overline{\lim_{r\to\infty}}\,(r^{-1}\ln|f(re^{i\theta}, z)|),\quad h_w(\theta) = \sup_{z \in C_z^n} h(\theta, z).$$

When $n > 1$ we may make the same remarks about the set N_w as we made above

1) In Ronkin's paper [1] Theorems 26.6 and 26.7 are formulated for the case $\rho = 1$. Their proof, however, carries over without the slightest essential change to the case of an entire function of arbitrary order.

about the sets M_σ and M_ρ.

In conclusion we note that the results we have set forth here can be carried over to the case when the growth of the entire function is specified by precise orders, and to the case (for $n > 1$) when $n - k$ of the variables are fixed, where $k > 1$, and the growth of the function is studied with respect to the remaining k variables.

4. Entire functions of finite degree.* An entire function $f(z)$, where $z \in C^n$, for which $\sigma_1, \cdots, \sigma_n$ constitute a conjugate system of types of order $(1, \cdots, 1)$ will be called a *function of finite degree*, and the numbers $\sigma_1, \cdots, \sigma_n$ will be called its *system of conjugate degrees*. By analogy with the case of a single variable, we express an entire function $f(z)$ of finite degree in the form of a series

$$f(z) = \sum_k \frac{c_k}{k!} z^k, \tag{5.57}$$

where, as usual, $z^k = z^{k_1} \cdots z^{k_n}$, $k! = k_1! \cdots k_n!$; $k_1, \cdots, k_n = 0, 1, 2, \cdots$.

Let us consider the series

$$F(z) = \sum_k \frac{c_k}{z^{k+1}} \left\{ \text{where } \frac{c_k}{z^{k+1}} = \frac{c_{k_1 \cdots k_n}}{z_1^{k_1+1} \cdots z_n^{k_n+1}} \right\}. \tag{5.58}$$

Let D be the domain of convergence of the series $\sum_k c_k \zeta^k$. As a result of the mapping $z_j = \zeta_j^{-1}$, $j = 1, \cdots, n$, and of completion by points at infinity, we obtain from it the domain $D \subset G_z^n$ (where G_z^n is the space of function theory) which is a neighborhood of the point $z = \infty$ (i.e., the point $z_1 = \cdots = z_n = \infty$). The series (5.58) converges uniformly in every domain B lying, together with its boundary, in the domain D. Therefore it defines in the domain D a holomorphic function $F(z)$. This is called the function *associated* to the function $f(z)$. If $\rho_1, \cdots, \rho_n$ are the conjugate radii of convergence of the series $\sum_k c_k \zeta^k$, then the quantities $r_j = 1/\rho_j$, $j = 1, \cdots, n$, are called the *conjugate radii of convergence of the series* (5.58).

The functions $f(z)$ and $F(z)$ are connected by the relationships

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Delta_{r+\epsilon}} F(\zeta)\, e^{\|z\zeta\|} d\zeta, \tag{5.59}$$

$$F(z) = e^{i\|\phi\|} \int_0^\infty \cdots \int_0^\infty f(te^{-i\phi}) \exp(-\|tze^{-i\phi}\|)\, dt. \tag{5.60}$$

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*Translator's note: In mathematical literature in the English language these functions are usually called "of exponential type".

Here and later $\Delta_{r+\epsilon}$ is the torus $\{|z_j| = r_j + \epsilon,\ j = 1, \cdots, n\}$ (where $r_1, \cdots, r_n$ are the conjugate radii of convergence of the series (5.58)) and, as usual, $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$, $dt = dt_1 \wedge \cdots \wedge dt_n$, $\|zt\| = z_1 t_1 + \cdots + z_n t_n$, $\|\phi\| = \phi_1 + \cdots + \phi_n$, $\|tze^{-i\phi}\| = t_1 z_1 e^{-i\phi_1} + \cdots + t_n z_n e^{-i\phi_n}$, $f(te^{-i\phi}) = f(t_1 z_1 e^{-i\phi_1}, \cdots, t_n z_n e^{-i\phi_n})$, and the regions of integration are taken to be oriented in a suitable way.

Equations (5.59) and (5.60) are generalizations of the corresponding formulas for functions of one variable.[1] They may be obtained by termwise integration of the power series representing the functions $f(z)$ and $F(z)$. Formula (5.60) holds in the region of uniform convergence of the integral on the right-hand side; this region depends on the choice of the quantities $\phi_1, \cdots, \phi_n$. If in some part of this region the series (5.58) diverges, this formula gives us an analytic extension of the function $F(z)$ beyond the boundary of the region D.

Just as in the case of one variable, there exists a close connection between the growth of the function $f(z)$ and the distribution of the singularities of the associated function $F(z)$. We have

THEOREM 26.9. *The hypersurface of conjugate degrees of the entire function $f(z)$ of finite degree coincides with the boundary of the image (in the absolute octant) of the domain of convergence of the power series representing the associated function $F(z)$ in the neighborhood of the point $z = \infty$.*

PROOF. We carry through the proof only for that part of the hypersurface of conjugate degrees which lies in the absolute octant. We make use of the fact that the conjugate radii of convergence of the series (5.58) are defined by the condition (see the corollary to Theorem 3.10; in our case $r^{-k} = r_1^{-k_1} \cdots r_n^{-k_n}$)

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k| r^{-k}} = 1. \tag{5.61}$$

On the other hand, by Theorem 26.2, the conjugate degrees $\sigma_1, \cdots, \sigma_n$ of the function $f(z)$ satisfy the condition

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k| \left(\frac{k}{e}\right)^k \frac{\sigma^{-k}}{k!}} = 1,$$

where

1) See, for example, Levin, *Entire functions*.

$$\left(\frac{k}{e}\right)^k \frac{\sigma^{-k}}{k!} = \left(\frac{k_1}{e}\right)^{k_1} \frac{\sigma_1^{-k_1}}{k_1!} \cdots \left(\frac{k_n}{e}\right)^{k_n} \frac{\sigma_n^{-k_n}}{k_n!}.$$

Accordingly

$$\overline{\lim_{\|k\|\to\infty}} \sqrt[\|k\|]{|c_k|\sigma^{-k}} = 1. \tag{5.62}$$

Comparing equations (5.61) and (5.62), we have Theorem 26.9.

There is an even closer connection between the growth of an entire function $f(z)$ of finite degree and the distribution of singularities of its associated function $F(z)$ than can be inferred from Theorem 26.9.

Suppose, as usual, that $z_j = r_j e^{i\phi_j}$, $j = 1, \cdots, n$. Let us consider those values of the real variables $\nu_1, \cdots, \nu_n$, or in other words, the point in the space R_n of the variables $\nu_1, \cdots, \nu_n$, for which we have, for fixed $\phi_1, \cdots, \phi_n$ and sufficiently large $r_1, \cdots, r_n$, asymptotically

$$|f(r_1 e^{i\phi_1}, \cdots, r_n e^{i\phi_n}| < e^{\|\nu r\|}.$$

Such points ν in the space R_n form a set which we denote by T_ϕ. It can be shown, just as for the analogous property of the set B_σ (see subsection 2 of this section), that the set T_ϕ is in fact a convex region. We denote by $T_{-\phi}$ the corresponding region defined by the quantities $-\phi_1, \cdots, -\phi_n$.

Let us now consider, in the same space R_n of the variables $\nu_1, \cdots, \nu_r$, the set C_ϕ of those points $(\nu_1, \cdots, \nu_n)$ for which the associated function $F(z)$ can be analytically extended in the domain

$$\{\mathrm{Re}\,(z_j e^{-i\phi_j}) > \nu_j,\ j = 1, \cdots, n\}.$$

For $n = 2$ V. K. Ivanov [1; 2], and for $n > 2$ M. Š. Stavskiĭ [1], have proved the following theorem, which is a generalization of the well-known theorem of Pólya on the connection between the indicator and the conjugate diagram of an entire function of one variable.[1)]

THEOREM 26.10. *For an entire function of finite degree the closed regions* $\overline{T}_{-\phi}$ *and* $\overline{C}_\phi$ *coincide.*

We omit the proof of this theorem.

REMARK. If in the space R_n we introduce spherical coordinates

1) See, for example, Levin, *Entire functions.*

$r, \alpha_1, \cdots, \alpha_{n-1}$ and write the equations of the boundaries of the regions T and C_ϕ in the form $r = h_T(\alpha_1, \cdots, \alpha_{n-1}, \phi_1, \cdots, \phi_n)$ and $r - h_C(\alpha_1, \cdots, \alpha_{n-1}, \phi_1, \cdots, \phi_n)$, respectively, then Theorem 26.10 can be phrased in another way:

THEOREM 26.10_1. *For an arbitrary entire function of finite degree*

$$h_T(\alpha_1, \cdots, \alpha_{n-1}, -\phi_1, \cdots, -\phi_n) = h_C(\alpha_1, \cdots, \alpha_{n-1}, \phi_1, \cdots, \phi_n).$$

5. **Application of the Fourier transform.** Among the entire functions of finite degree a special place belongs to those functions $f(z)$ which for real values of the variables $z_j = x_j + iy_j$ $(j = 1, \cdots, n)$ belong to the class L^2 throughout the whole space R_n^x of the real variable $x_1, \cdots, x_n$ (a function $f(x) \in L^p$ in the space R_n^x if the quantity $|f(x)|^p$ is measurable in the sense of Lebesgue in this space and if $\int_{R_n^x} |f(x)|^p dx < \infty$). The characteristics of the growth of such functions are closely connected with the properties of their Fourier transforms. In the case of functions of one variable this connection is expressed in the well-known Paley-Wiener theorem, as follows.[1)]

Let the function $f(x)$ be of class L^2 on the entire x-axis. Then in order that its Fourier transform

$$\tilde{f}(t) = \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{itx}\, dt \tag{5.63}$$

vanish almost everywhere outside some bounded interval, it is necessary and sufficient that the function $f(x)$ be extendable from the real axis to the entire plane of the complex variable $z = x + iy$ as an entire function of finite degree. In this connection, the smallest interval (a, b) outside of which $\tilde{f}(t) \equiv 0$ is defined by the conditions

$$a = -\overline{\lim_{r \to \infty}} \{r^{-1} \ln |f(-ir)|\},$$

$$b = \overline{\lim_{r \to \infty}} \{r^{-1} \ln |f(ir)|\}.$$

REMARK. The symbol l.i.m. (limit in mean) in equation (5.63) means that

$$\lim_{T_1, T_2 \to \infty} \int_{-\infty}^{\infty} |\tilde{f}(t) - \frac{1}{2\pi} \int_{-T_1}^{T_2} f(x)\, e^{-itx}\, dt\,|^2\, dt = 0.$$

This symbol will be used in a similar sense below.

In order to formulate the corresponding theorem for the case of $n > 1$ vari-

1) See, for example, Levin, *Entire functions*.

ables, we consider, following Pólya and Plancherel [1], the characteristic of the growth of the function $f(z)$ in different directions in the space R_n^y of the variables $y_1, \cdots, y_n$. Let α_j be the angle made by the chosen direction with the coordinate axis Oy_j $(j = 1, \cdots, n)$. We write

$$h_f(\alpha, x) = \overline{\lim_{r\to\infty}} \{r^{-1} \ln |f(x_1 + ir\cos\alpha_1, \cdots, x_n + ir\cos\alpha_n)|\}.$$

The function

$$h_f(\alpha) = \sup_{x\in R_n^x} h_f(\alpha, x)$$

will be called the *P-indicator* of the function $f(z)$. The following theorem, obtained by M. Plancherel and G. Pólya [1], is a generalization of the Paley-Wiener theorem to the case of $n > 1$ variables.

THEOREM 26.11 (Plancherel-Pólya). *Let the function $f(x)$ belong to the class L^2 in the entire space of the real variables $x_1, \cdots, x_n$. Then in order that its Fourier transform*

$$\tilde{f}(t) = \text{l.i.m.} \left[\frac{1}{\sqrt{2\pi}}\right]^n \int_{R_n^{(x)}} f(x)\, e^{i\|tx\|}\, dx \tag{5.64}$$

vanish almost everywhere outside some bounded region, it is necessary and sufficient that the function $f(x)$ can be extended to the whole space C^n of the complex variables $z_j = x_j + iy_j$ $(j = 1, \cdots, n)$ as an entire function of finite degree. In this connection, the smallest bounded convex region D_f outside of which $\tilde{f}(t) \equiv 0$ is defined by the equation

$$K_f(\alpha) = h_f(\alpha).$$

Here $K_f(\alpha)$ is the support function of the region D_f and $h_f(\alpha)$ is the P-indicator of the entire function $f(z)$ of finite degree which is the extension of the given function $\tilde{f}(t)$.

PROOF. 1) Necessity. Suppose $\tilde{f}(t) \equiv 0$ almost everywhere outside some bounded convex region D_f with the support function $K_f(\alpha)$. We consider the function

$$f(z) = \left[\frac{1}{\sqrt{2\pi}}\right]^n \int_{R_n^{(t)}} \tilde{f}(t)\, e^{-i\|tz\|}\, dt. \tag{5.65}$$

It is the desired extension of the function $f(x)$ given in the space of the real variables $x_1, \cdots, x_n$ to the whole space C^n of the complex variables $z_1, \cdots, z_n$. We show that it has the properties stated in the theorem.

In fact, in view of the known properties of the Fourier transform of a function of class L^2, the function $\tilde{f} \in L^2$. From this, and from the boundedness of the region D_f, it follows that $\tilde{f} \in L^1$; accordingly, the integral in equation (5.65) can be taken in the usual sense.

From the fact that the region over which the integral (5.65) is taken is bounded, it follows also that the integral converges absolutely and uniformly in an arbitrary bounded region of the space C^n. Therefore $f(z)$ is an entire function.

Let us now prove that $f(z)$ is an entire function of finite degree, and estimate its P-indicator. We have

$$|f(x_1 + ir\cos\alpha_1, \cdots, x_n + ir\cos\alpha_n)| =$$

$$= \frac{1}{(\sqrt{2\pi})^n}\left|\int_{D_f} \tilde{f}(t)\exp\{i\|tx\| + r\|t\cos\alpha\|\}\,dt\right| \leq$$

$$\leq \left[\frac{1}{\sqrt{2\pi}}\right]^n \int_{D_f} |\tilde{f}(t)| \exp\{r\|t\cos\alpha\|\}\,dt \leq$$

$$\leq M\exp\{r\sup_{t\in D_f}\|t\cos\alpha\|\} = Me^{rK_f(\alpha)}. \tag{5.66}$$

It follows that $f(z)$ is an entire function of finite degree and its P-indicator satisfies the condition

$$h_f(\alpha) \leq K_f(\alpha). \tag{5.67}$$

The necessity has been proved.

2) Sufficiency. Let $f(z)$ be an entire function of finite degree, belonging to the class L^2 for real values of the independent variables $z_j = x_j + iy_j$ $(j = 1, \cdots, n)$

We assume to begin with that $f(x) \in L^1$. Then the Fourier transform $\tilde{f}(t)$ of the function $f(x)$ exists not only as a l. i. m., but in the ordinary sense, i.e.,

$$\tilde{f}(t) = \frac{1}{(\sqrt{2\pi})^n}\int_{R_n^{(x)}} f(x)\,e^{i\|tx\|}\,dx. \tag{5.68}$$

We select in the space $R_n^{(x)}$ some line passing through the origin of coordinates and introduce on it a new coordinate system $x_1', \cdots, x_n'$ in which this line plays the role of the x_1'-axis. Then

$$x_p = \sum_{q=1}^{n} x_q' \cos\alpha_{pq}, \quad p = 1, \cdots, n,$$

where α_{pq} is the angle between the axes of x_q' and x_p. The change of variables defined by these formulas converts the function $f(x)$ into the function

$$g(x') = f(a_1 + x_1' \cos\alpha_1, \cdots, a_n + x_1' \cos\alpha_n),$$

where

$$a_p = \sum_{q=2}^{n} x_q' \cos\alpha_{pq}, \quad p = 1, \cdots, n.$$

It is obvious that, together with the function $f(x)$, we have also $g(x') \in L^1$. If in the integral $\int_{R_n} |g(x')|\, dx'$ we pass to repeated integrals and make use of Fubini's theorem, we find that $\int_{-\infty}^{\infty} |g(x')|\, dx_1'$ exists for almost all $(x_2', \cdots, x_n') \in R_{n-1}$.

We carry out this change of variables in the integral (5.68) and then separate out the integration with respect to x_1'. As a result we obtain

$$\tilde{f}(t) = \frac{1}{(\sqrt{2\pi})^n} \int_{R_n^{(x')}} g(x') \exp\left\{i \sum_{k=1}^{n} \left[t_k \sum_{q=1}^{n} x_q' \cos\alpha_{kq}\right]\right\} dx' =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{R_{n-1}^{(x_2', \cdots, x_n')}} e^{i\|ta\|}\, dx' \left[\int_{-\infty}^{\infty} g(x') \times \right.$$

$$\left. \times \exp\left\{ix_1' \sum_{k=1}^{n} t_k \cos\alpha_{k,1}\right\} dx_1'\right]. \tag{5.69}$$

Here the inner integral exists for almost all points $(x_2', \cdots, x_n') \in R_{n-1}$ in the outer integral, and $dx' = dx_2' \wedge \cdots \wedge dx_n'$. It is obvious that

$$g(x') = f(a_1 + z\cos\alpha_{11}, \cdots, a_n + z\cos\alpha_{n1})|_{y=0},$$

where f is an entire function of finite degree of the complex variable $z = x_1' + iy$. In view of the definition of the P-indicator of a function of one variable,

$$\overline{\lim_{r\to\infty}} \{r^{-1} \ln |f(a_1 + ir\cos\alpha_{11}, \cdots, a_n + ir\cos\alpha_{n1})| \le h_f(\alpha_{11}, \cdots, \alpha_{n1})\}.$$

It follows from the Paley-Wiener theorem that for

$$\sum_{k=1}^{n} t_k \cos\alpha_{k,1} \ge h_f(\alpha_{11}, \cdots, \alpha_{n1}) \tag{5.70}$$

and for almost all points $(x_2', \cdots, x_n') \in R_{n-1}$

$$\int_{-\infty}^{\infty} g(x') \exp\left\{ix_1' \sum_{k=1}^{n} t_k \cos\alpha_{k1}\right\} dx_1' = 0.$$

Moreover, taking into account equation (5.69), we conclude that the function $\tilde{f}(t) \equiv 0$ almost everywhere in the half-space defined by the condition (5.70). Reasoning in this way, we show that in the case under discussion the function $\tilde{f}(t) \equiv 0$ almost everywhere outside some bounded convex region, with a support function satisfying the condition

$$K_f(\alpha) \geq h_f(\alpha). \tag{5.71}$$

Let us now consider the case when $f(x) \in L^2$ but $f(x) \notin L^1$. We write

$$f_\epsilon(z) = f(z) \prod_{k=1}^{n} \frac{\sin \epsilon z_k}{\epsilon z_k}.$$

It is easy to see that the function $f_\epsilon(x) \in L^1$; by what we have just proved, the Fourier transform $\tilde{f}_\epsilon(t)$ of this function vanishes almost everywhere outside some convex domain D_ϵ with the support function $K_{f_\epsilon}(\alpha)$ satisfying the condition $K_{f_\epsilon}(\alpha) \leq h_f(\alpha) + \epsilon$.

We show that $\lim_{\epsilon \to 0} \tilde{f}_\epsilon(t) = \tilde{f}(t)$ for almost all points $t \in R_n^{(t)}$. In fact, the Fourier transform of the function $(\sin \epsilon x)/\epsilon x$ is equal to zero for $|t| > 1$ and equal to $(2\epsilon)^{-1}$ for $|t| < 1$. Taking into account the fact that the Fourier transform of the product of two functions is equal to the convolution of the Fourier transforms of the factors, we find

$$\tilde{f}_\epsilon(t) = \frac{1}{(\sqrt{2\pi})^n} \int_{R_n^{(x)}} f_\epsilon(x) e^{i\| tx \|} dx = \frac{1}{(2\epsilon)^n} \int_{S(t,\epsilon)} \tilde{f}(\tau)\, d\tau.$$

Here $S(t, \epsilon) \{ |\tau_j - t_j| < \epsilon, \ j = 1, \cdots, n\} \subset R_n^{(\tau)}$.

It is easy to see that for almost all points $t \in R_n^t$

$$\lim_{\epsilon \to 0} \frac{1}{(2\epsilon)^n} \int_{S(t,\epsilon)} \tilde{f}(\tau)\, d\tau = \tilde{f}(t),$$

and accordingly that

$$\lim_{\epsilon \to 0} \tilde{f}_\epsilon(t) = \tilde{f}(t).$$

But the function $\tilde{f}_\epsilon(t) \equiv 0$ almost everywhere outside the domain D_ϵ. Accordingly we have also $\tilde{f}(t) \equiv 0$ almost everywhere outside some convex domain D_f, of which the support function satisfies the condition

$$K_f(\alpha) \leq h_f(\alpha). \tag{5.72}$$

Comparing equations (5.71) and (5.72) we see that

$$K_f(\alpha) = h_f(\alpha).$$

With this the theorem is proved.

Fourier transforms may also be successfully applied to the study of the growth of functions holomorphic in radial tubular domains.[1)] For these, in particular, there exists a theorem analogous to the Pólya-Plancherel Theorem 26.11.

1) See, for example, Bochner-Martin [1].

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